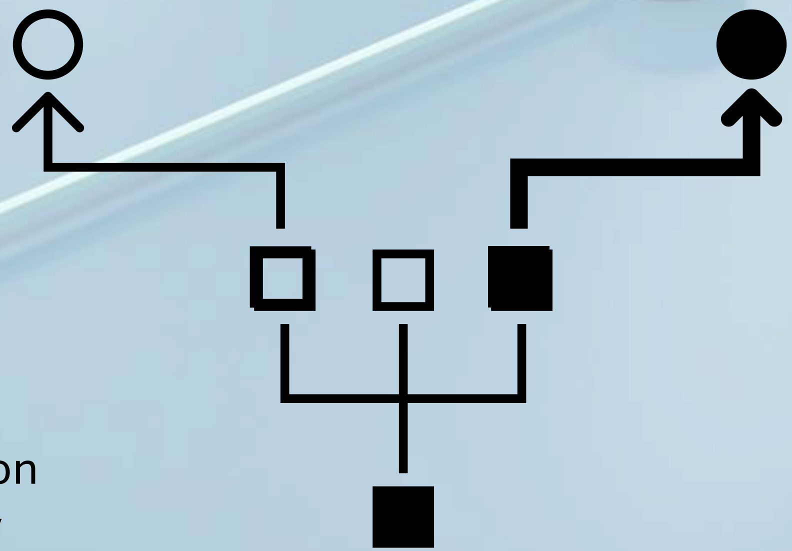
Perfect Chiral Quantum Routing



- **QSQW2025 -** 11th International conference on Quantum Simulation and Quantum Walks, Naples, Italy

Author: Simone Cavazzoni

- **Università degli Studi di Modena e Reggio Emilia**

Quantum Walk Formalism

The Chiral Hamiltonian:

Graph G : G(V,E)

- ***V set of vertices.***
- ***E set of Edges.***

$$A_{jk} = A^{*}_{kj} = \begin{cases} \beta e^{i\phi_{kj}} & if\ j\ , k\ connected \\ 0 & otherwise \end{cases}$$

$$e^{-iAt}|\psi_0\rangle = |\psi_f\rangle$$

The perfect Router:

© $|\psi_0\rangle$: ***input state***

© $|\psi_{f,l}\rangle$: **output state(s)**

$$\langle\psi_0|\psi_{f,l}\rangle = 0$$

$$\langle\psi_{f,l}|\psi_{f,m}\rangle = \delta_{\ l,m}$$

The Lily Graph

The Structure:

- Output layer
- Routing layer
- Chiral layer
- Input layer

Hamiltonian Representation:

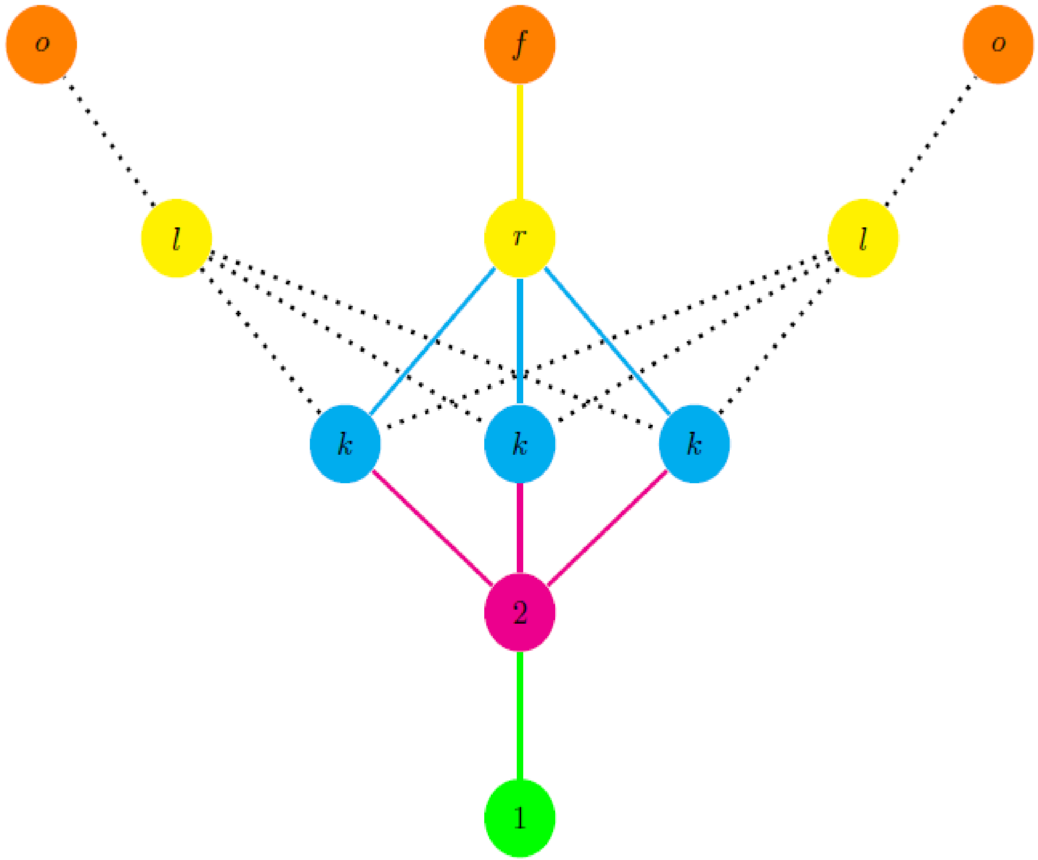
- $H(n, d, \vec{\phi}) = A^{in} + A^{C}(d, \vec{\phi}) + A^{R}(n, d, \vec{\phi}) + A^{out}(n)$

  - $A^{in} = |1\rangle\langle 2| + |2\rangle\langle 1|$

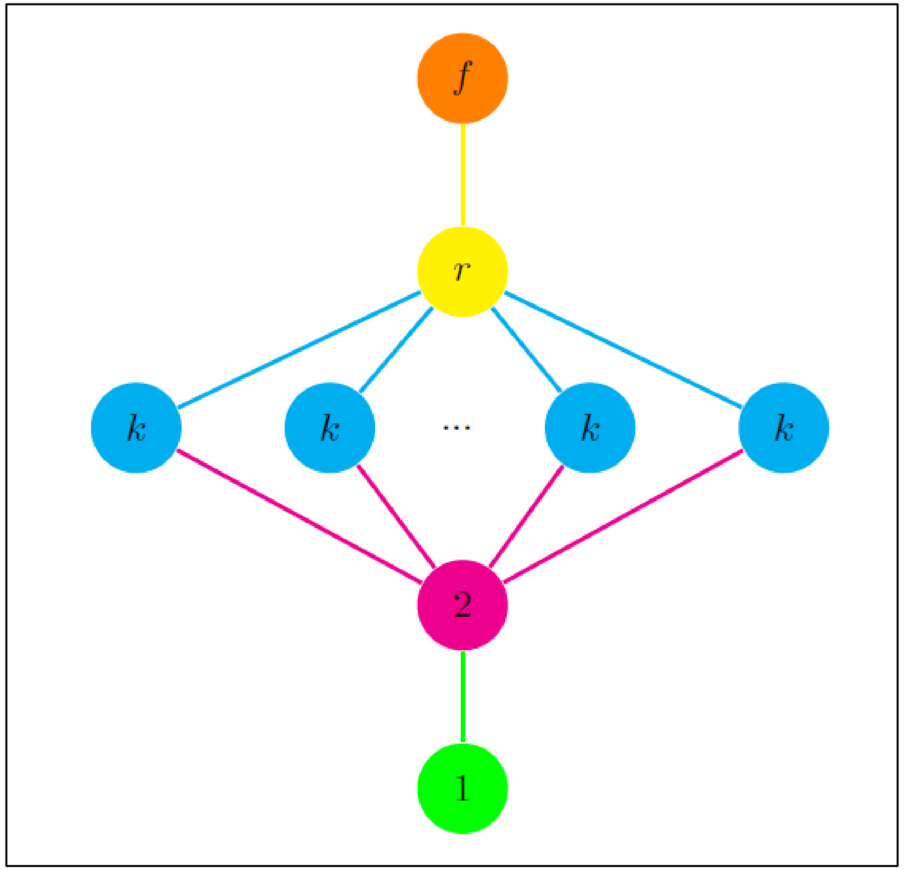
  - $A^{C}(d, \vec{\phi}) = \sum_{k\epsilon C} e^{-i\phi_k} |2\rangle\langle k| + e^{i\phi_k}|k\rangle\langle 2|$

  - $A^{R}(n, d, \vec{\phi}) = \sum_{k\epsilon C} \sum_{l\epsilon R, l\neq r} |k\rangle\langle l| + |l\rangle\langle k| + \sum_{k\epsilon C} e^{-i\phi_k} |r\rangle\langle k| + e^{i\phi_k}|k\rangle\langle r|$

  - $A^{out}(n) = \sum_{a\epsilon R} \sum_{b\epsilon O} |a\rangle\langle b| + |b\rangle\langle a|$



The Effective Topology



- The root of unity $\sqrt[d]{1}$:

$$\sum_{k=1}^{d} e^{-i\phi_k} = \sum_{k=1}^{d} e^{-i\frac{2\pi k}{d}} = 0$$

- Constructive interference in $|r\rangle$
- Distructive interference in $|l\rangle$

- The dynamics of the walker involves only the desired output(s)

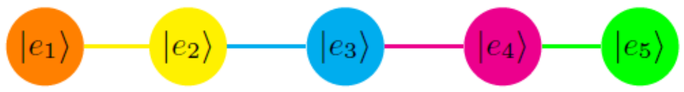
The Dimensionality Reduction Method

The Krylov Reduction Method:

- $\langle\zeta|e^{-iHt}|\psi_0\rangle = \sum_{k=0}^{\infty}\frac{(-it)^k}{k!}\langle\zeta|H^k|\psi_0\rangle = \langle\zeta|e^{-iH_{red}t}|\psi_{0-red}\rangle$

- $\mathcal{J}(H,|\zeta\rangle) = span\left(\{H^k|\zeta\rangle | k \in \mathbb{N}_0\}\right)$

Krylov Representation



- **Reduced Basis:**
  - $|e_1\rangle = |f\rangle$
  - $|e_2\rangle = |r\rangle$
  - $|e_3\rangle = \frac{1}{\sqrt{d}}\sum_C e^{i\phi_k}|k\rangle$
  - $|e_4\rangle = |2\rangle$
  - $|e_5\rangle = |1\rangle$

- **Basis of identically evolving vertices**

- $H_{red} = \langle e_j|H|e_k\rangle$

The Purely Topological Routing Process

$$H_{red}(d) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{d} & 0 & 0 \\ 0 & \sqrt{d} & 0 & \sqrt{d} & 0 \\ 0 & 0 & \sqrt{d} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- **Input State:** $|\psi_0\rangle = \alpha|1\rangle + e^{i\gamma}\sqrt{1-\alpha^2}\,|2\rangle$
  - $\alpha = 0\,;\ \alpha = 1$ **routing of classical information**
- **Output State:** $|\psi_f\rangle = \alpha|f\rangle + e^{i\gamma}\sqrt{1-\alpha^2}|r\rangle$

$$P_{1f}(t) = \frac{\left(2d - (2d+1)\cos(t) + \cos(t\sqrt{1+2d})\right)^2}{4(1+2d)^2}$$

$$P_{2r}(t) = \frac{\left(\cos(t) - \cos(t\sqrt{1+2d})\right)^2}{4}$$

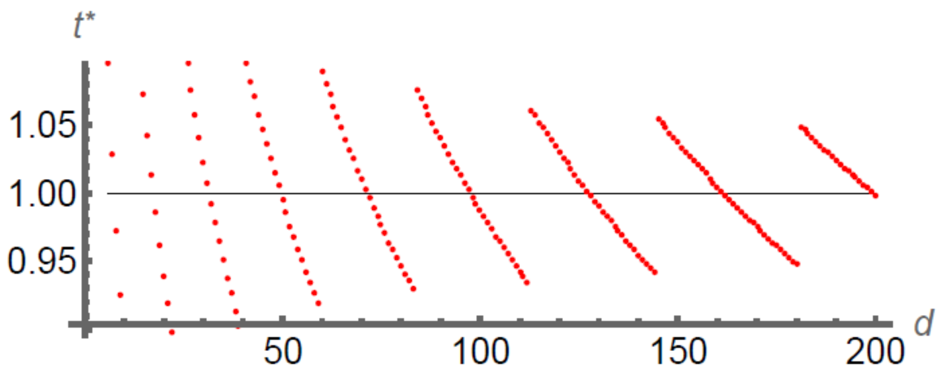


Fig: Optimal time t^* (in unit of π) maximizing the routing probability $P_{1f}(t^*)$ as a function of the chiral layer dimension d.

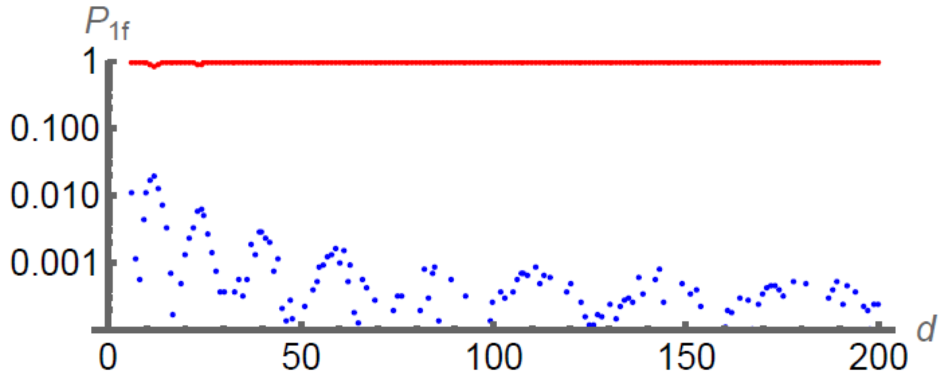


Fig: Logplot of the maximized routing probability $P_{1f}(t^*)$ as a function of the chiral layer dimension d (red). The difference $|P_{1f}(t^*) - P_{2r}(t^*)|$ (blue).

$$©\, P_{1f}(t_d^* \approx \pi) \approx 1$$
$$©\, P_{2r}(t_d^* \approx \pi) \approx 1$$

→ **Nearly optimal Chiral Quantum routing**

The Weighted Routing Process

- $H\left(n, d, \vec{\phi}\right) = A^{in} + \beta A^{C}\left(d, \vec{\phi}\right) + \beta A^{R}\left(n, d, \vec{\phi}\right) + A^{out}(n)$

$$H_{red,\beta}(\beta, d) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \beta\sqrt{d} & 0 & 0 \\ 0 & \beta\sqrt{d} & 0 & \beta\sqrt{d} & 0 \\ 0 & 0 & \beta\sqrt{d} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- **Input State:** $|\psi_0\rangle = \alpha|1\rangle + e^{i\gamma}\sqrt{1-\alpha^2}|2\rangle$
- **Output State:** $|\psi_f\rangle = \alpha|f\rangle + e^{i\gamma}\sqrt{1-\alpha^2}|r\rangle$

- $\alpha = 0\,;\ \alpha = 1$ **routing of classical information**

$$P_{1f}(t) = \frac{\left(2d\beta^2 - (2d+\beta^2)\cos(t) + \cos\left(t\sqrt{\beta^2+2d}\right)\right)^2}{4(\beta^2+2d)^2}$$

$$P_{2r}(t) = \frac{\left(\cos(t) - \cos\left(t\sqrt{\beta^2+2d}\right)\right)^2}{4}$$

© $P_{1f}\left(t^{*}_{\beta\sqrt{d}=\sqrt{3}/\sqrt{2}} = \pi\right) = 1$

© $P_{2r}\left(t^{*}_{\beta\sqrt{d}=\sqrt{3}/\sqrt{2}} = \pi\right) = 1$

Perfect Chiral Quantum routing

Robustness & Time Periodicity

Taylor Expansion of the Probabilities:

$$P_{1f}(t) = 1 - (t - t^*_\beta)^2 + O(t - t^*_\beta)^3$$
$$P_{2r}(t) = 1 - \frac{5}{2}(t - t^*_\beta)^2 + O(t - t^*_\beta)^3$$

Reduced Basis:

- $|e_1\rangle = |f\rangle$
- $|e_2\rangle = |r\rangle$
- $|e_3\rangle = \frac{1}{\sqrt{d}} \sum_C e^{i\phi_k} |k\rangle$
- $|e_4\rangle = |2\rangle$
- $|e_5\rangle = |1\rangle$

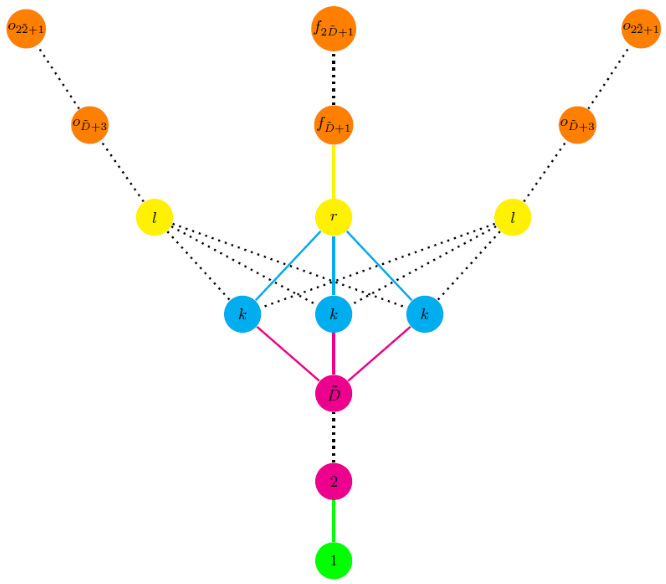
$$e^{-iH_{red,\beta}(2q+1)\pi} = \sum_{s=0}^{4} |e_{1+s}\rangle\langle e_{5-s}|, \ q \in \mathbb{N}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

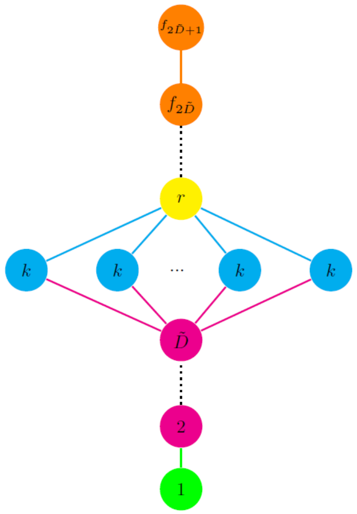
Perfect Periodic Chiral Quantum routing

Extension to Qudit Routing and Physical Implementations

$$H = \frac{1}{2} \sum_{j \in Lily} J_j \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + \sum_{j \in Lily} B_j \sigma_j^z$$



$$\widetilde{H} = \frac{1}{2} \sum_{j \in K.\ Lily} \widetilde{J}_j \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + \sum_{j \in K.\ Lily} \widetilde{B}_j \sigma_j^z$$



$$\begin{cases} \widetilde{J}_j = \sqrt{j(\tilde{D} - j)}, \\ \widetilde{B}_j = \frac{(\widetilde{J}_{j-1} + \widetilde{J}_j)}{2} - \frac{1}{2(2\tilde{D}-1)} \sum_{l=1}^{2\tilde{D}-1} \widetilde{J}_l \end{cases}$$

Reduced Basis:

$|e_1\rangle = |f_{2\tilde{D}+1}\rangle$, $|e_2\rangle = |f_{2\tilde{D}}\rangle$

$\dots$

$|e_{\tilde{D}-1}\rangle = |f_1\rangle$, $|e_{\tilde{D}}\rangle = |r\rangle$

$|e_{\tilde{D}+1}\rangle = \frac{1}{\sqrt{d}} \sum_{k\in\mathcal{C}} e^{i\phi_k} |k\rangle$

$|e_{\tilde{D}}+2\rangle = |\tilde{D}\rangle$, $|e_{\tilde{D}+3}\rangle = |\tilde{D}-1\rangle$

$\dots$

$|e_{2\tilde{D}}\rangle = |2\rangle$, $|e_{2\tilde{D}+1}\rangle = |1\rangle$

$$e^{-i\tilde{H}\,(2q+1)\pi} = \sum_{s=0}^{2\tilde{D}} |e_{1+s}\rangle\langle e_{2\tilde{D}+1-s}|,\ q \in \mathbb{N}$$

$$= \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & 0 \\ 0 & 1 & \iddots & \iddots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Input State: $|\psi_0\rangle = \sum_{l=1}^{\tilde{D}} \alpha_l |l\rangle$

Output State: $|\psi_f\rangle = \alpha_{\tilde{D}}|r\rangle + \sum_{l=0}^{\tilde{D}-1} \alpha_l |f_{2\tilde{D}+1-l}\rangle$

Perfect Periodic Chiral Quantum routing

Summary & Conclusions

- © Nearly perfect quantum routing only with topology and chirality
- © Perfect Chiral quantum routing with the weighted configuration

© Both for quantum and classical Information (bit, qubit and qudit)

© Universality of the time required promote our result to the most general routing procedure