Search and state transfer between hubs by quantum walks

Martin Štefaňák and Stanislav Skoupý

QSQW 2025, 14. 1. 2025

Outline

1. Introduction
2. State transfer between hubs in CTQW
3. Search for hubs in DTQW
4. State transfer between two hubs in DTQW
5. State transfer between multiple hubs in DTQW

Outline

Introduction

Hub = fully connected vertex (degree $N-1$)

Razzoli, Bordone & Paris, JPA **55**, 265303 (2022)

- Continuous-time quantum walk (CTQW) on otherwise arbitrary graph with marked hubs
- Universal behaviour of probability amplitudes at the hubs
- Applications for spatial search and quantum transport
- Search for single or multiple hubs behaves like on the complete graph

Our results

- Utilization of search for state transfer in CTQW
- Extension to discrete-time quantum walk (DTQW)
- Search — with some modification, reduction to complete graph
- State transfer between unique sender and receiver — works for 2 OG states
- State transfer between multiple sender and receiver vertices

Outline

State transfer between hubs in CTQW

- Simple graph $G = (V, E)$, N vertices
- Hilbert space — N-dimensional complex vector space spanned by $|v\rangle$, $v \in V$
- W — set of marked hubs
- Hamiltonian — weighted Laplacian with additional potential on the marked vertices

$$\hat{H} = \gamma \hat{L} + \sum_{w \in W} \lambda_w |w\rangle\langle w|$$

- Optimal search — hopping rate $\gamma = \frac{1}{N}$, weights of marked hubs $\lambda_w = -1$
- State transfer — split W into set of senders $\mathcal{S}$ and set of receivers $\mathcal{R}$

$$|\mathcal{S}| = S, \quad |\mathcal{R}| = R, \quad S + R = M$$

- Keep the same hopping rate and weights

State transfer between hubs in CTQW

- 3D invariant subspace

$$|g\rangle = \frac{1}{\sqrt{N-M}} \sum_{v \notin W} |v\rangle, \quad |\mathcal{S}\rangle = \frac{1}{\sqrt{S}} \sum_{s \in \mathcal{S}} |s\rangle, \quad |\mathcal{R}\rangle = \frac{1}{\sqrt{R}} \sum_{r \in \mathcal{R}} |r\rangle$$

- Effective hamiltonian in the invariant subspace

$$H = -\frac{1}{N} \begin{pmatrix} S & \sqrt{RS} & \sqrt{S(N-M)} \\ \sqrt{RS} & R & \sqrt{R(N-M)} \\ \sqrt{S(N-M)} & \sqrt{R(N-M)} & -M \end{pmatrix}$$

- Energy spectrum — $E_0 = 0, \quad E_{\pm} = \pm\sqrt{\frac{M}{N}}$

State transfer between hubs in CTQW

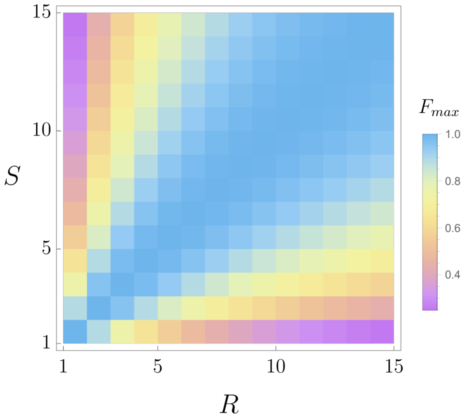
- Initial state $|\mathcal{S}\rangle$, target state $|\mathcal{R}\rangle$
- Fidelity of state transfer

$$\begin{aligned} F(t) &= |\langle\mathcal{R}|e^{-i\hat{H}t}|\mathcal{S}\rangle|^2 \\ &= \frac{4RS}{M^2}\sin^4\left(\frac{E_+ t}{2}\right) + \frac{RS}{NM}\sin^2(E_+ t) \end{aligned}$$

- Maximal fidelity for $T = \pi/E_+ = \pi\sqrt{\frac{N}{R+S}}$

$$F_{max} = \frac{4RS}{(R+S)^2}$$

- We achieve high fidelity state transfer for $S \approx R$



Outline

Description of DTQW

- Simple graph $G = (V, E)$
- Hilbert space — direct sum of local spaces $\mathcal{H}_v$

$$\mathcal{H} = \bigoplus_{v\in V} \mathcal{H}_v, \quad \mathcal{H}_v = \mathrm{Span}\left\{|v,v\rangle, |v,w\rangle | (v,w)\in E\right\}, \quad \dim \mathcal{H}_v = d_v + 1$$

- Evolution operator — coin and shift — $\hat{U} = \hat{S}\hat{C}$
- Flip-flop shift operator — $\hat{S}|v,w\rangle = |w,v\rangle, \quad \hat{S}|v,v\rangle = |v,v\rangle$
- Coin acts locally on $\mathcal{H}_v$ — $\hat{C} = \bigoplus_v \hat{C}_v$
- We consider Grover operator with a weighted loop

$$\hat{G}_v(\ell_v) = 2|\Omega_v(\ell_v)\rangle\langle\Omega_v(\ell_v)| - \hat{I}_v, \quad |\Omega_v(\ell_v)\rangle = \frac{1}{\sqrt{d_v + \ell_v}}\left(\sqrt{d_v}|\Omega_v\rangle + \sqrt{\ell_v}|v,v\rangle\right)$$

$$|\Omega_v\rangle = \frac{1}{\sqrt{d_v}} \sum_{\substack{w \\ \{v,w\}\in E}} |v,w\rangle$$

Search for hubs in DTQW

- Solutions of search — hubs from set $\mathcal{M}$, $|\mathcal{M}| = M$
- Tune local weights according to the degrees — $\ell_v = N - d_v$, $\ell_m = 1$
- Marking of solutions — phase shift of π
- Evolution operator for DTQW search

$$\hat{U}_{\mathcal{M}} = \hat{S}\left(\bigoplus_{v\notin\mathcal{M}}\left(\hat{G}_v(N-d_v)\right)\bigoplus_{m\in\mathcal{M}}(-\hat{G}_m(1))\right)$$

- Usual choice of initial state — the equal weight superposition of all arcs

$$|\tilde{\Omega}\rangle = \frac{1}{\sqrt{\sum\limits_{v\in V} d_v}}\sum_{v\in V}\sqrt{d_v}|\Omega_v\rangle$$

- Does not work, it is not an eigenstate of unperturbed evolution (without π phase shift)

Search for hubs in DTQW

- Modification of the initial state — equal distribution into vertex subspaces

$$|\Omega\rangle = \frac{1}{\sqrt{N}} \sum_{v\in V} |\Omega_v(N-d_v)\rangle$$

- 5D invariant subspace, 4D if there is only one marked hub
- Problem reduces to search on a complete graph with loops
- Probability of detecting one of the marked vertices

$$P(2t) = P(2t+1) = \sin^2\left(\frac{\omega(2t+1)}{2}\right), \quad \omega = \arccos\left(1-\frac{2M}{N}\right)$$

- Success probability close to 1 after T steps

$$T = \frac{\pi}{\omega} \approx \frac{\pi}{2}\sqrt{\frac{N}{M}} + O\left(\sqrt{\frac{M}{N}}\right)$$

Outline

State transfer between two hubs

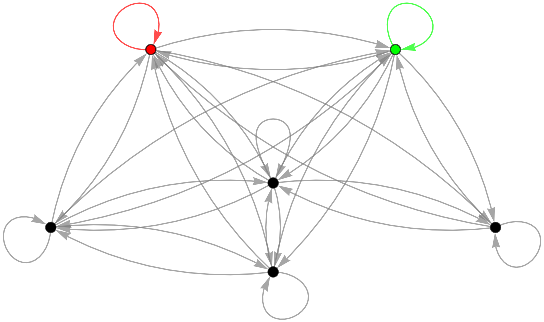
Two choices for initial and target states

- Loops at sender and receiver

$$|s, s\rangle, \quad |r, r\rangle$$

- Superpositions of arcs

$$|\Omega_s\rangle, \quad |\Omega_r\rangle$$



State transfer between two hubs

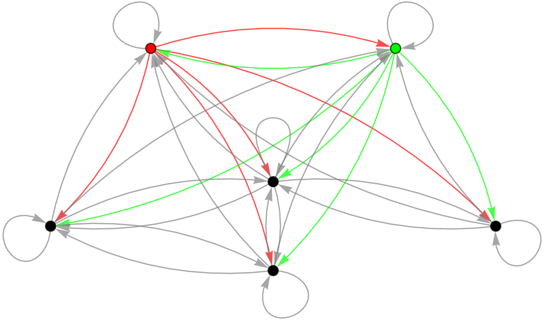
Two choices for initial and target states

- Loops at sender and receiver

$$|s, s\rangle, \quad |r, r\rangle$$

- Superpositions of arcs

$$|\Omega_s\rangle, \quad |\Omega_r\rangle$$



State transfer between two hubs

- Evolution operator for two marked vertices

$$\hat{U}_{s,r} = \hat{S}\left(\bigoplus_{v\neq s,r}\left(\hat{G}_v(N-d_v)\right)\oplus(-\hat{G}_s(1))\oplus(-\hat{G}_r(1))\right)$$

- 9D exact invariant subspace

$$|\nu_1\rangle = |s,s\rangle, \qquad |\nu_2\rangle = |r,r\rangle, \qquad |\nu_3\rangle = |s,r\rangle, \qquad |\nu_4\rangle = |r,s\rangle$$

$$|\nu_5\rangle = \frac{1}{\sqrt{N-2}}\sum_{v\neq s,r}|s,v\rangle, \qquad |\nu_6\rangle = \frac{1}{\sqrt{N-2}}\sum_{v\neq s,r}|r,v\rangle$$

$$|\nu_7\rangle = \frac{1}{\sqrt{N-2}}\sum_{v\neq s,r}|v,s\rangle, \qquad |\nu_8\rangle = \frac{1}{\sqrt{N-2}}\sum_{v\neq s,r}|v,r\rangle$$

$$|\nu_9\rangle = \frac{1}{N-2}\sum_{v\neq s,r}\left(\sqrt{N}|\Omega_v(N-d_v)\rangle - |v,r\rangle - |v,s\rangle\right)$$

State transfer between two hubs

- Decomposition into symmetric and antisymmetric subspace — $\hat{U} = \hat{U}_+ \oplus \hat{U}_-$
- 5D symmetric subspace

$$\begin{aligned} |\sigma_1\rangle &= \frac{1}{\sqrt{2}}\left(|\nu_1\rangle + |\nu_2\rangle\right), \quad |\sigma_2\rangle = \frac{1}{\sqrt{2}}\left(|\nu_3\rangle + |\nu_4\rangle\right), \quad |\sigma_3\rangle = \frac{1}{\sqrt{2}}\left(|\nu_5\rangle + |\nu_6\rangle\right) \\ |\sigma_4\rangle &= \frac{1}{\sqrt{2}}\left(|\nu_7\rangle + |\nu_8\rangle\right), \quad |\sigma_5\rangle = |\nu_9\rangle \end{aligned}$$

- 4D antisymmetric subspace

$$\begin{aligned} |\tau_1\rangle &= \frac{1}{\sqrt{2}}\left(|\nu_1\rangle - |\nu_2\rangle\right), \quad |\tau_2\rangle = \frac{1}{\sqrt{2}}\left(|\nu_3\rangle - |\nu_4\rangle\right) \\ |\tau_3\rangle &= \frac{1}{\sqrt{2}}\left(|\nu_5\rangle - |\nu_6\rangle\right), \quad |\tau_4\rangle = \frac{1}{\sqrt{2}}\left(|\nu_7\rangle - |\nu_8\rangle\right) \end{aligned}$$

State transfer between two hubs

- Evolution in the symmetric subspace — eigenvalues ± 1 and $\lambda_1^{(\pm)} = e^{\pm i\omega_1}$

$$\omega_1 = \arccos\left(1 - \frac{4}{N}\right)$$

- Evolution in the antisymmetric subspace — $\lambda_2^{(\pm)} = e^{\pm i\omega_2}$ and $\lambda_3^{(\pm)} = e^{\pm i\omega_3}$

$$\begin{aligned} \omega_2 &= \arccos\left(\sqrt{1 - \frac{2}{N}}\right) = \frac{\omega_1}{2} \\ \omega_3 &= \arccos\left(-\sqrt{1 - \frac{2}{N}}\right) = \pi - \omega_2 \end{aligned}$$

Transfer of the loop state

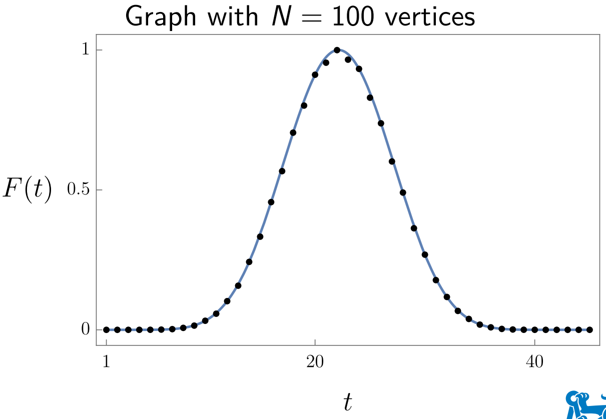
- For large N, loop states $|s,s\rangle$, $|r,r\rangle$ are superpositions of $|1\rangle$, $|\pm\omega_1\rangle$ and $|\pm\omega_2\rangle$
- π rotation in $|\pm\omega_2\rangle$ plane, 2π rotation in $|\pm\omega_1\rangle$ plane

- Straightforward since $\omega_1 = 2\omega_2$
- Fidelity of state transfer

$$F(t) = \sin^8\left(\frac{\omega_2 t}{2}\right)$$

- High fidelity state transfer for

$$T = \frac{\pi}{\omega_2} \approx \pi\sqrt{\frac{N}{2}}$$



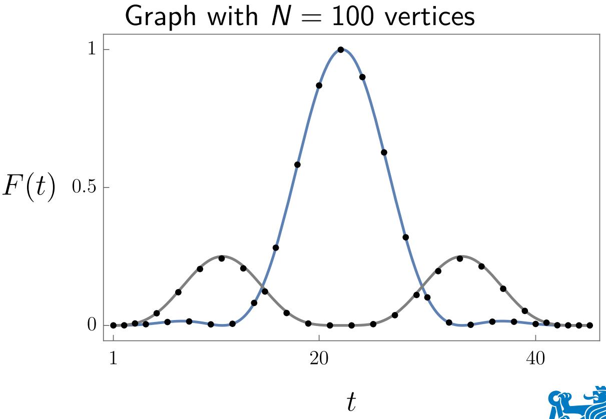
Transfer of the superposition of outgoing arcs

- For large N, $|\Omega_s\rangle$ and $|\Omega_r\rangle$ are superpositions of $|-1\rangle$, $|\pm\omega_1\rangle$, $|\pm\omega_2\rangle$ and $|\pm\omega_3\rangle$
- π rotation in $|\pm\omega_2\rangle$ and $|\pm\omega_3\rangle$ planes, 2π rotation in $|\pm\omega_1\rangle$ plane

- $\omega_3 = \pi - \omega_2$ — rapid oscillations between odd and even steps
- Fidelity of state transfer at even steps

$$F(t) = \cos^2(\omega_2 t)\sin^4\left(\frac{\omega_2 t}{2}\right)$$

- Same run-time, T even



Transfer of arbitrary qubit state

- Transfer of $|s,s\rangle \to |r,r\rangle$ and $|\Omega_s\rangle \to |\Omega_r\rangle$ happens in the same run-time
- We can transfer an arbitrary state of a qubit

$$|\psi_s\rangle = \alpha|s,s\rangle + \beta|\Omega_s\rangle \xrightarrow{\hat{U}^T_{s,r}} |\psi_r\rangle = \alpha|r,r\rangle + \beta|\Omega_r\rangle$$

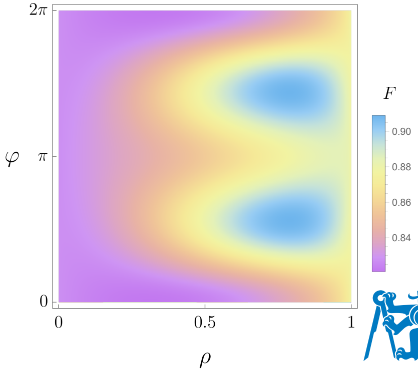
- For finite N fidelity depends on the initial state

$$|\psi_s\rangle = \rho|s,s\rangle + e^{i\varphi}\sqrt{1-\rho^2}|\Omega_s\rangle$$

- For all states fidelity behaves as

$$F(N) = 1 - O\left(N^{-1}\right)$$

Graph with $N = 10$ vertices



Outline

State transfer between multiple hubs in DTQW

- S sender vertices from set $\mathcal{S}$, R receiver vertices from set $\mathcal{R}$
- Invariant subspace has dimension 11
- Splitting into symmetric/antisymmetric subspace not possible unless $S = R$
- Spectrum of $\hat{U}_{\mathcal{S},\mathcal{R}}$ has similar form as for $\hat{U}_{s,r}$
- Eigenvalues are ± 1 and three conjugated pairs $\lambda_j^{(\pm)} = e^{\pm i\omega_j}$

$$\omega_1 = \arccos\left(1 - \frac{2(R+S)}{N}\right), \quad \omega_2 = \frac{\omega_1}{2}, \quad \omega_3 = \pi - \omega_2$$

Transfer from superposition of loops

- Initial state — superposition of loops at senders

$$|\phi_\ell\rangle = \frac{1}{\sqrt{S}}\sum_{s\in\mathcal{S}}|s,s\rangle$$

- Probability of transferring to any basis state at any receiver vertex

$$P_\ell(t) = \frac{4R}{(R+S)^2}\sin^4\left(\frac{\omega_2 t}{2}\right)$$

- High probability only for $S = R = 1$

Transfer from superposition of outgoing arcs

- Initial state — superposition of all outgoing arcs from all senders

$$|\Omega_{\mathcal{S}}\rangle = \frac{1}{\sqrt{S}} \sum_{s \in \mathcal{S}} |\Omega_s\rangle$$

- Total transfer probability in even steps

$$P_\Omega(2t) = \frac{4RS}{(R+S)^2} \sin^4(\omega_2 t)$$

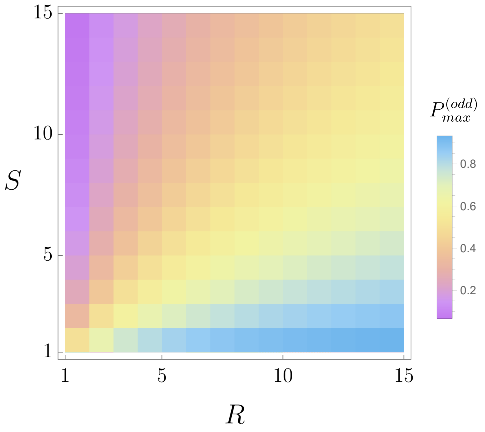
- High probability of state transfer for $S \approx R$
- Same result as for the CTQW

Transfer from superposition of outgoing arcs

- Total transfer probability in odd steps

$$P_\Omega(2t+1) = \frac{R}{R+S}\sin^2(\omega_2(2t+1))$$

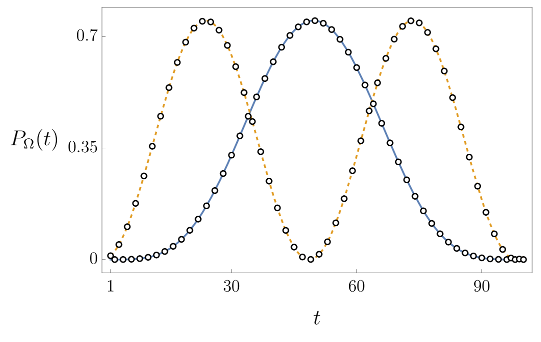
- High probability of state transfer for $R \gg S$
- This regime does not exist in CTQW



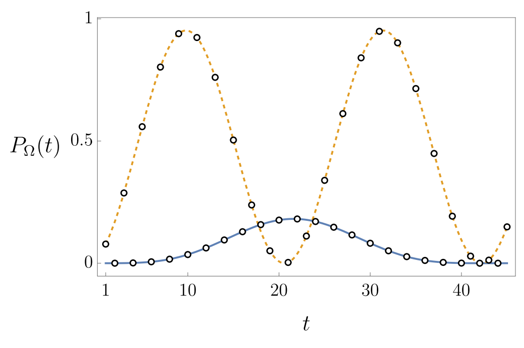
Transfer from superposition of outgoing arcs

- State transfer on a graph with $N = 1000$ vertices from a single sender

$S = 1,\ R = 3$



$S = 1,\ R = 20$



References

- L. Razzoli, P. Bordone, and M. G. A. Paris, Universality of the fully connected vertex in laplacian continuous time quantum walk problems, J. Phys. A: Math. Theor. **55**, 265303 (2022)
- S. Skoupy and M. Stefanak, Search and state transfer between hubs by quantum walks, Phys. Rev. A **110**, 022422 (2024)

Thank you for your attention