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Classification of One-dimensional Single Fermionic Quantum Cellular Automata

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PLAN OF THE TALK

- **<u>Quantum Cellular Automata</u>**
- **<u>Fermionic Cellular Automata (FCA)</u>**
- **<u>Main result 1°: Circuital Implementability</u>**
- **<u>Main result 2°: Classification of 1-D FCA</u>**

QUANTUM CELLULAR AUTOMATA

[1] B. Schumacher and R. F. Werner. “Reversible Quantum Cellular automata”. arXiv:quant-ph/0405174 (2004).

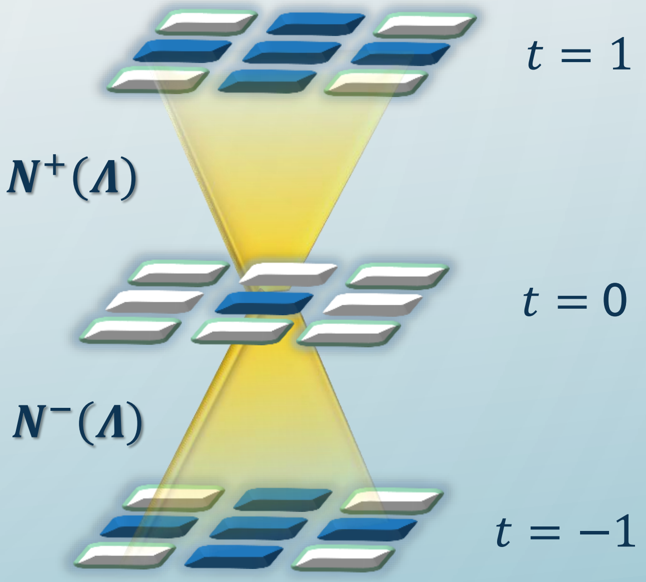
Quantum Cellular Automata (**QCA**) are the quantum version of **Cellular Automata**. The space of states spans a **finite-dimensional Hilbert space** $\mathcal{H}_x$ such that $\mathcal{U}\colon \bigotimes_x \mathcal{H}_x \rightarrow \bigotimes_x \mathcal{H}_x$ is a unitary operator describing the discrete-time evolution

Heisenberg picture!

The action of α is assigned on the **algebra of observables** $\mathcal{A}(\mathcal{L})$ on an infinite lattice $\mathcal{L} \subseteq \mathbb{Z}^s$ [1]

$$\boldsymbol{\alpha}\colon \boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{L}}) \rightarrow \boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{L}})$$

- $\alpha(A_1 A_2) = \alpha(A_1)\alpha(A_2)$ — *automorphism*
- $A_\Lambda \in \mathcal{A}_\Lambda \Rightarrow \alpha(A_\Lambda) \subset \mathcal{A}(\Lambda + N^+(\Lambda))$ — *locality*
- $\alpha \circ \tau_x = \tau_x \circ \alpha,\ \tau_y(\mathcal{A}_x) = \mathcal{A}_{y+x}$ ***shift automaton*** — *translational invariant*
- $\alpha(\mathcal{A}_x) = \alpha(\mathcal{A}_o)$ — *local transition rule*



Λ:

$\Lambda + N^+(\Lambda)$:

$\Delta\Lambda = (\Lambda + N^+(\Lambda))/\Lambda$:

Applications

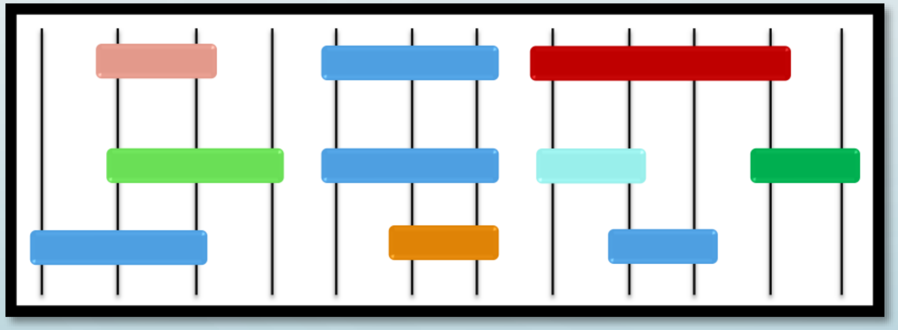
- *Quantum foundations, computation, and simulation*
- *Quantum field theories and topological phases of matter*

EXAMPLES OF QCA

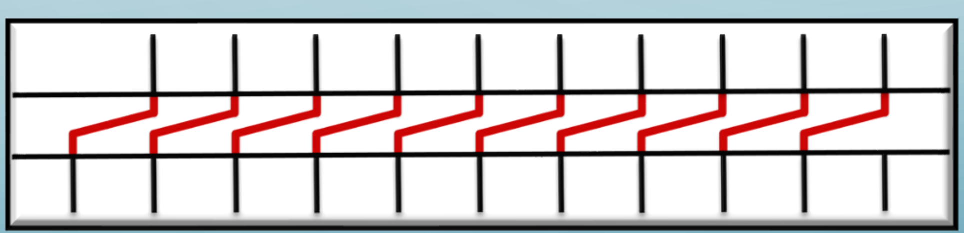
- Finite lattice, with finite dimensional systems at each site

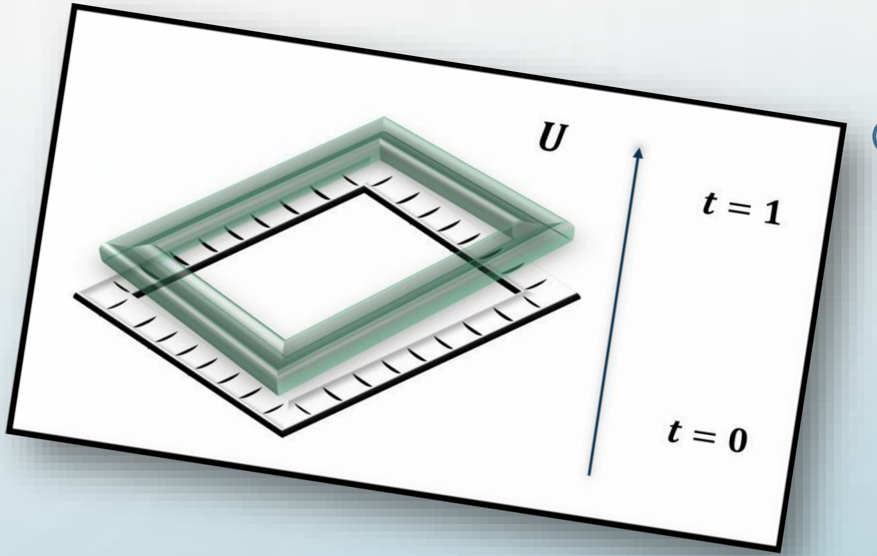
$$\alpha(O) = U^{\dagger}(O \otimes I)\, U \qquad U \textit{ unitary matrix}$$

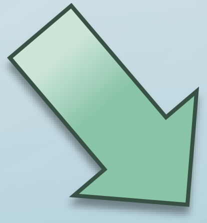
- **Quantum circuits** are QCA...



- ... But **not all** QCA are quantum circuits! (Ex: *Shift* τ)







- How many automata may you find? (classification)
- Are automata finite depth quantum circuit (FDQC)? (circuital implementability)

FERMIONIC CELLULAR AUTOMATA

[2] S. Bravyi and A. Kitaev. "Fermionic quantum computation". Annals of Physics, **298** (2002).

- **Fermionic theories** ⇒ **Fermionic (CAR) Algebra** for finitely many **Local Fermionic Modes** arranged on $\mathcal{L} \subseteq \mathbb{Z}$ [2].

$$\{\phi_x, \phi_y^\dagger\} = \delta_{xy} I, \qquad \{\phi_x, \phi_x\} = \{\phi_y, \phi_y\} = 0, \qquad \forall\, x, y \in \mathcal{L}.$$

- Represent the CAR of Local Fermionic Modes by assigning the pair of odd operators (X_x, Y_x) to each site $x \in \mathcal{L}$

$$X_x = \frac{\phi_x + \phi_x^\dagger}{\sqrt{2}}, \qquad Y_x = \frac{-i(\phi_x - \phi_x^\dagger)}{\sqrt{2}}, \qquad Z_x = iY_x X_x = \frac{\phi_x^\dagger \phi_x - \phi_x \phi_x^\dagger}{2}$$

Fermionic Pauli matrices

- *Parity Superselection Rule* ⇒ CAR algebra is a **$\mathbb{Z}_2$-graded algebra of observables** $\mathcal{A} = (\mathcal{A}^0, \mathcal{A}^1)$
  - *$A \otimes B$ is replaced by $A \boxtimes B$ (**graded tensor product**)*
  - $\{[O_1, O_2]\} = O_1 O_2 - (-1)^{\deg O_1 \deg O_2} O_2 O_1$ (***graded commutator***)

Fermionic Cellular Automata (FCA) are **automorphisms** $\mathcal{T}$: $\mathcal{A}(\mathbb{Z}) \to \mathcal{A}(\mathbb{Z})$ of the **$\mathbb{Z}_2$-graded algebra** $\mathcal{A}(\mathbb{Z})$ which **preserve the parity** of its elements: $\{\mathcal{T}(\xi_x), \mathcal{T}(\eta_x)\} = 0$ for all (ξ_x, η_x) odd generators of CAR

[3] D. Gross et al. "Index theory of one dimensional quantum walks and cellular automata". Commun. Math. Phys. **310**, 419-454 (2012).
[4] L. Fidkowski et al. "Interacting invariants for Floquet phases of fermions in two dimensions". Phys. Rev. B **99**, 085115 (2019).

INDEX THEORY FOR QCA AND FCA [3,4]

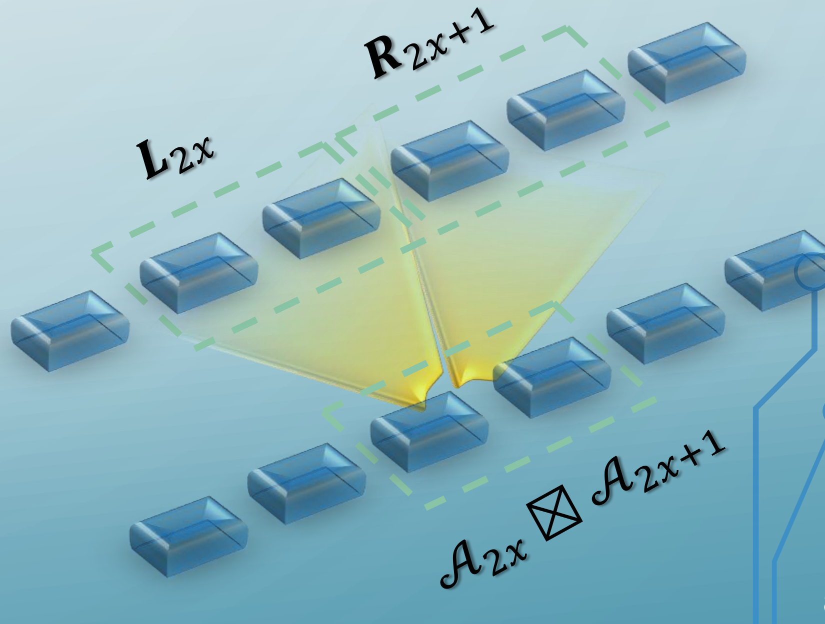
- The Left and Right **Support Algebras** $\boldsymbol{L}_{2x},\ \boldsymbol{R}_{2x}$ describe how many information "*moves left and right*"

$$\boldsymbol{L}_{2x} = \boldsymbol{S}(\mathcal{T}(\mathcal{A}_{2x} \boxtimes \mathcal{A}_{2x+1}), \mathcal{A}_{2x-1} \boxtimes \mathcal{A}_{2x}) \qquad \boldsymbol{R}_{2x+1} = \boldsymbol{S}(\mathcal{T}(\mathcal{A}_{2x} \boxtimes \mathcal{A}_{2x+1}), \mathcal{A}_{2x} \boxtimes \mathcal{A}_{2x+1})$$

$$\mathcal{T}(\mathcal{A}_{2x} \boxtimes \mathcal{A}_{2x+1}) = \boldsymbol{L}_{2x} \boxtimes \boldsymbol{R}_{2x+1}$$

- **Index of a QCA (FCA)** is a local invariant that quantifies the net flow of "*quantum information*"

$$ind(\mathcal{T}) = \sqrt{\frac{\dim(\boldsymbol{L}_{2x})}{\dim(\mathcal{A}_{2x})}} = \sqrt{\frac{\dim(\mathcal{A}_{2x+1})}{\dim(\boldsymbol{R}_{2x+1})}}$$



- **Index-one QCA** are **FDQC!**

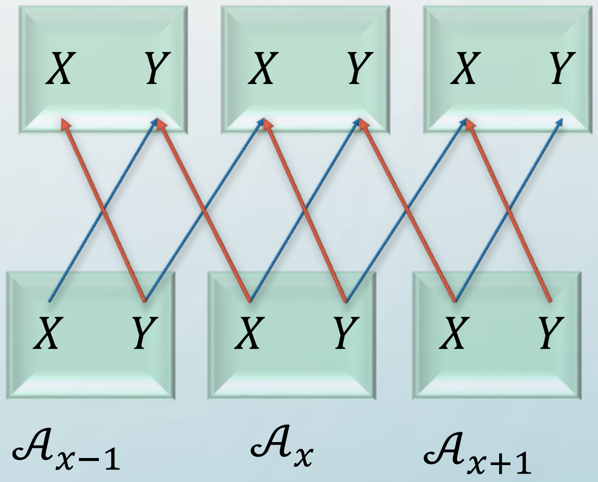
EXAMPLES OF FCA AND FERMIONIC INDEX

- **Conjugation:** $SU(2) = \left(SU^0(2), SU^1(2)\right)$ special unitary group

$$\boldsymbol{\mathcal{U}(A_x) = U^\dagger(A_x \boxtimes I)U, \qquad U \in SU^{0,1}(2)}$$

- **Majorana shift (Fermionic translation):**

$$\sigma_+ : (X_x, Y_x) \mapsto (Y_x, X_{x+1})$$
$$\sigma_-^{-1} : (X_x, Y_x) \mapsto (Y_{x-1}, X_x)$$



Index of FCA

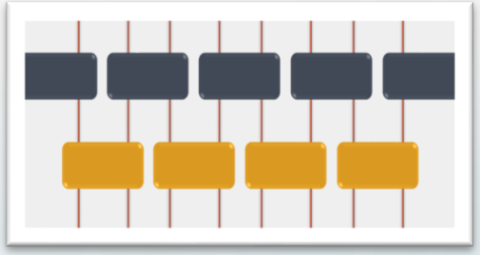
Given a support algebra $\mathcal{S}_x$,

$$ind(\mathcal{T}) = \begin{cases} \frac{p+q}{d} \in \mathbb{Q}, & \mathcal{S}_x \simeq Mat\left(\mathbb{C}^{p|q}\right) \\ \sqrt{2}\,\frac{p+q}{d} \in \mathbb{R}, & \mathcal{S}_x \simeq \mathcal{C}\ell_1(p|q) \end{cases}$$

Examples: $ind(I) = 1,\ ind\left(\tau_x^{\pm}\right) = d^{\pm 1},\ \ \dim \mathcal{H}_x = d,\ ind\left(\sigma_{\pm}\right) = 2^{\pm\frac{1}{2}}$

[4] L. Fidkowski et al. “Interacting invariants for Floquet phases of fermions in two dimensions”. Phys. Rev. B **99**, 085115 (2019).

CIRCUITAL IMPLEMENTABILITY FOR FCA



Fermionic Circuits

- **Finite Depth Fermionic Circuit** (**FDFC $\mathcal{F}$**) as Fermionic counterpart of FDQC: sequence of layers of unitary transformations acting over a finite set of cells in a partition of the lattice
- **Margolus Partition Scheme** (**MS $\mathcal{M}$**): FDFC with *depth* $D = 2$ and *nearest neighborhood scheme*
- If $\mathcal{T} = \mathcal{F}$, then $\mathcal{T}$ is ***$\mathcal{F}$-implementable.*** If $\mathcal{T} = \mathcal{M}$, then $\mathcal{T}$ is ***$\mathcal{M}$-implementable (locally implementable)***

[4] Equivalences

- ***$\mathcal{F}$-Equivalence****:* $\mathcal{T}' = \mathcal{F} \circ \mathcal{T}$
- ***Stable Equivalence:*** $\mathcal{T}' \boxtimes \boldsymbol{I} = \mathcal{F} \circ (\mathcal{T} \boxtimes \boldsymbol{I})$, where $\boldsymbol{I}$ is the identity automorphism over a set of ancillae
- If $ind(\mathcal{T}) = ind(\mathcal{T}')$, then $\mathcal{T}, \mathcal{T}'$ are **stably equivalent**, in particular ***$\mathcal{M}$-stably equivalent on supercells***
- Every **index-one FCA** is ***$\mathcal{M}$-implementable*** *over the enlarged system of ancillae:* $\mathcal{T} \boxtimes \boldsymbol{I} = \mathcal{M}$

MAIN RESULTS

[5] M. Freedman et al. “The Group Structure of Quantum Cellular Automata”. Commun. Math. Phys. 389, 1277-1302 (2022).

NEW!

ANCILLA REMOVAL FOR FCA

Ancilla Removal [5] $\Longrightarrow$ **No ancillas are needed to implement index-one FCA as quantum circuits!**

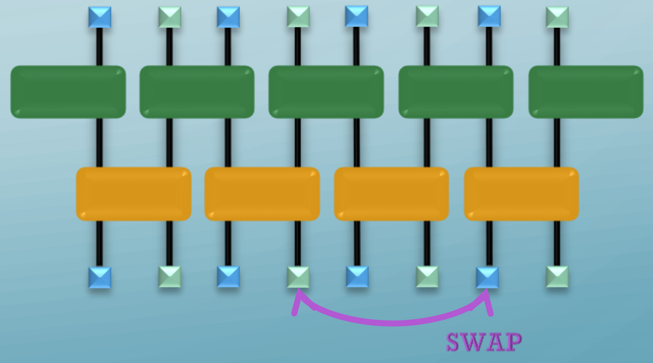
✓ Every **index-one FCA** satisfies ***$\mathcal{F}$-implementability*** *($\mathcal{T} = \mathcal{F}$)*

✓ If $ind(\mathcal{T}) = ind(\mathcal{T}')$, then $\mathcal{T}' = \mathcal{F} \circ \mathcal{T}$

✓ Upon **regrouping of cells**, $\mathcal{F}$ may be recast in $\mathcal{M}$

Ancilla Removal

- **“Borrowing”**: replacing the action on an ancilla with the same action on a sufficiently distant physical cell
- The circuit $\mathcal{U}^{\dagger}\mathcal{V}$ acts trivially on the ancillae $\Rightarrow$ physical cells **keep the same state** as before the updating



$$\mathcal{T}' \boxtimes I = \mathcal{U}^{\dagger}\mathcal{V} \circ (\mathcal{T} \boxtimes I)$$

$$\mathcal{T}' = \mathcal{F} \circ \mathcal{T}$$

NEW!

CLASSIFICATION OF ONE-DIMENSIONAL FCA

1° Step:

Classify FCA on **one-dimensional lattice** of a **single Local Fermionic Mode** cells isomorphic to $\mathrm{Mat}\left(\mathbb{C}^{1|1}\right)$

- The generators of the Local Fermionic Mode located in x are (X_x, Y_x)
- $\mathcal{T}$ is a **nearest neighbor FCA**: $\mathcal{T}(\mathcal{A}_x) \subseteq \mathcal{E}_L \boxtimes \mathcal{E}_C \boxtimes \mathcal{E}_R$
- If $\mathcal{A}_x \simeq Mat\left(\mathbb{C}^{1|1}\right)$, then $ind(\mathcal{T}) = 1, 2^{\pm\frac{1}{2}}, 2^{\pm 1}$

1. Every FCA $\mathcal{T}$ with $\boldsymbol{ind(\mathcal{T}) = 2^{\pm 1}}$ is of the form $\boldsymbol{\mathcal{T} = \tau_x^{\pm} \circ S}$
2. Every FCA $\mathcal{T}$ with $\boldsymbol{ind(\mathcal{T}) = 2^{\pm\frac{1}{2}}}$ is of the form $\boldsymbol{\mathcal{T} = \sigma_{\pm} \circ S}$

$ind(\boldsymbol{S}) = 1$

Classify all **index-one FCA**

2° Step: Classify **index-one** FCA on one-dimensional lattice of a single Local Fermionic Mode cells isomorphic to $Mat(\mathbb{C}^{1|1})$

Case 1. $\mathcal{E}_L \simeq \mathcal{E}_R \simeq I$

Conjugation $\mathcal{U}(\xi)$

Case 2. $\mathcal{E}_L \simeq \mathcal{E}_R$ generated by an **even operator**

Graded counterpart of the Schumacher-Werner qubit automaton

$$I \boxtimes \mathcal{T}_0(\xi) \boxtimes I = \mathcal{G}_\phi^\dagger (I \boxtimes \mathcal{U}(\xi) \boxtimes I)\mathcal{G}_\phi$$

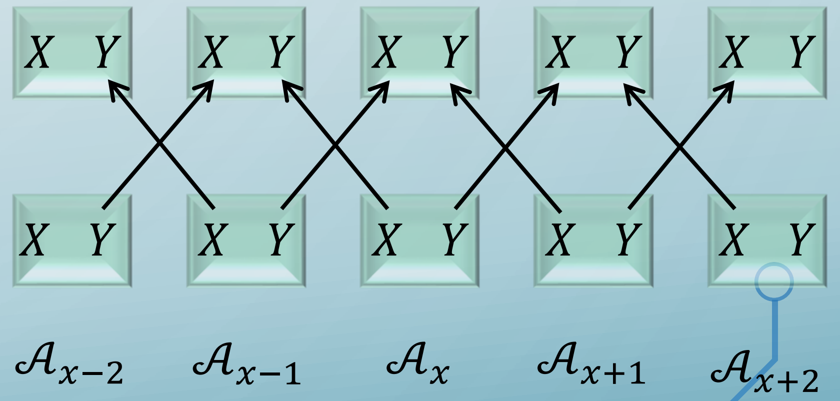
$$\mathcal{G}_\phi = (C_\phi \boxtimes I_3)(I_1 \boxtimes C_\phi), \quad C_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix}$$

Case 3. $\mathcal{E}_L \simeq \mathcal{E}_R$ generated by an **odd operator**

Forking Automaton

New FCA!

$$\tilde{\mathcal{T}}_0 : (X_x, Y_x) \mapsto (Y_{x-1}, X_{x+1})$$



Local implementability of 1-D FCA

Every **index-one FCA** over $Mat(\mathbb{C}^{1|1})$ is *$\mathcal{M}$-implementable*: $\mathcal{T} = \mathcal{M}$

SUMMARY & FUTURE OUTLOOKS

- ✓ **Fermionic Cellular Automata (FCA)**
- ✓ **Circuital implementability of FCA**
- ✓ **Equivalence of FCA**
- ✓ **Classification of 1-D FCA**

- **Classification of more general FCA**
- **Renormalization of FCA**
- **Thermal Theory for QCA**
- **...**

Thanks for the attention!