UNIVERSITÀ DELLA CALABRIA
DIPARTIMENTO DI FISICA

Interplay between topology and disorder in the eSSH chain

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E. G. Cinnirella, A. Nava, G. Campagnano, and D. Giuliano, Phys. Rev. B 109, 035114 (2024)

Summary

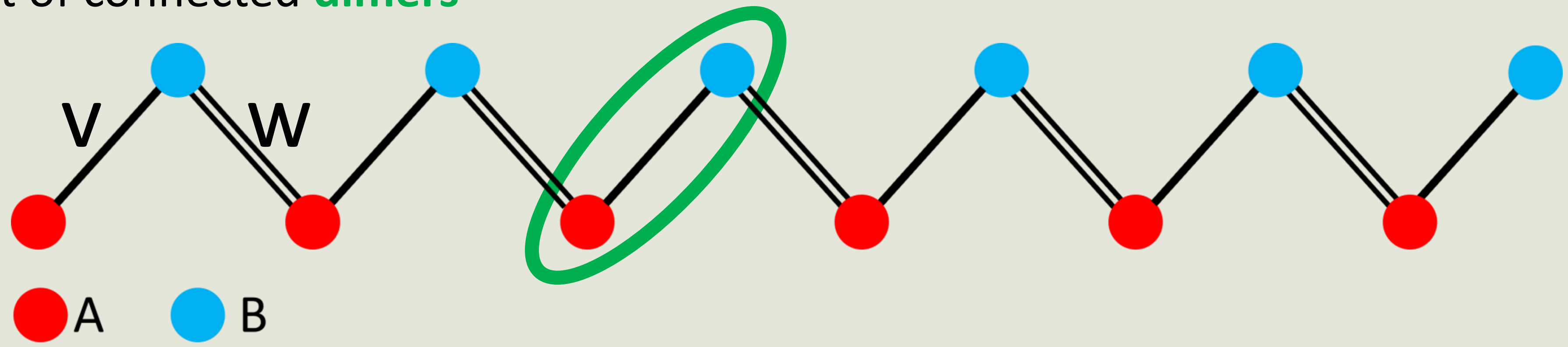
- Introduction
  - SSH/eSSH models
  - Winding Number
  - Lindblad Master Equation & EOD
- Disordered eSSH
  - Definition of disorder
  - Computational steps
  - Numerical results
  - Discussion of results
- Conclusions
- *Long Range Kitaev*

Introduction

- SSH & eSSH model
- Winding Number
- Lindblad Master Equation & EOD

SSH model

The SSH model describes a chain with staggered next-neighbour hoppings, defining a set of connected **dimers**

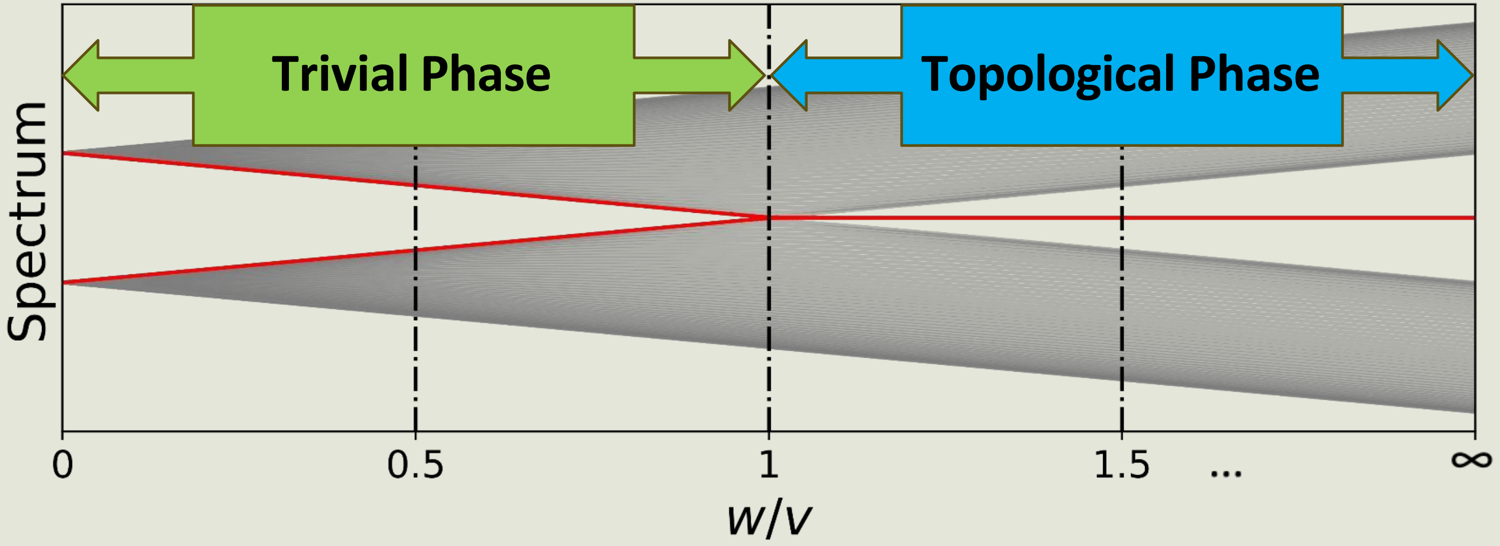


The Hamiltonian is quadratic in the fermionic operators and contains hopping terms only:

$$H_{SSH} = \sum_i \left(v c^+_{A,j} c_{B,j} + w c^+_{B,j} c_{A,j+1} \right) + h.c.$$

SSH model

The SSH spectrum is composed by two bands, symmetric around E=0. For **w>v** the spectrum contains two zero-energy modes.



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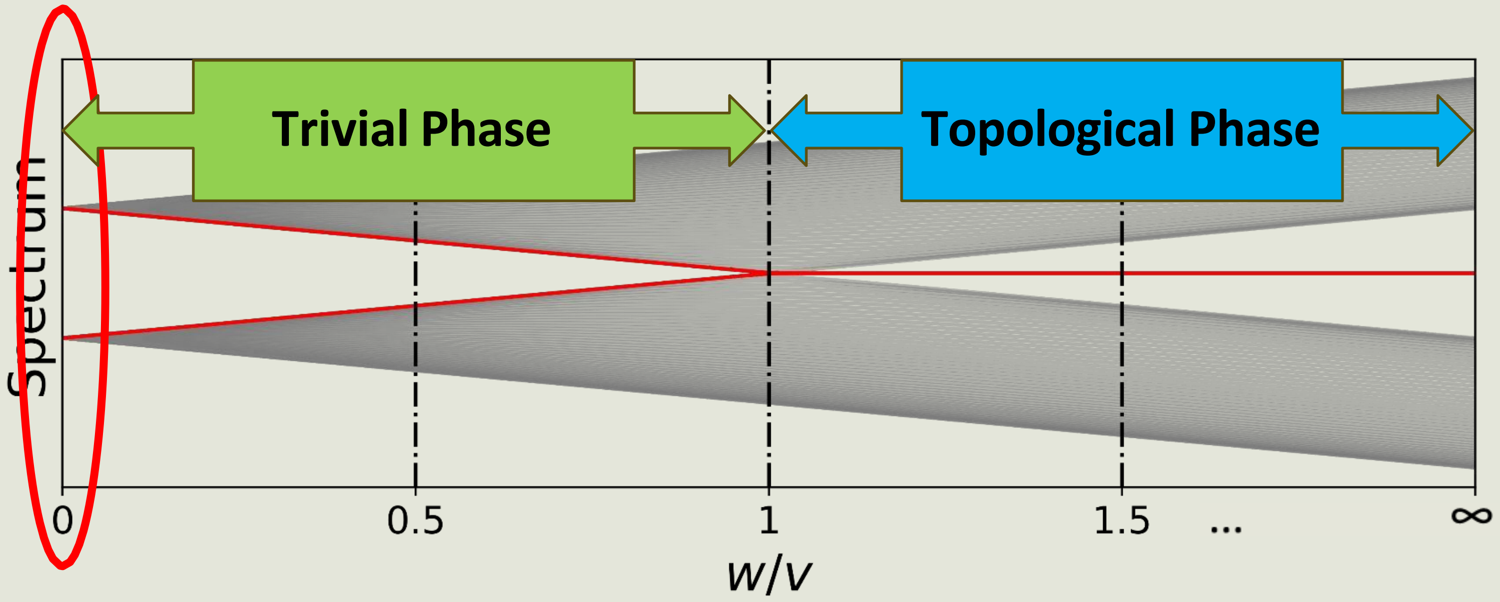

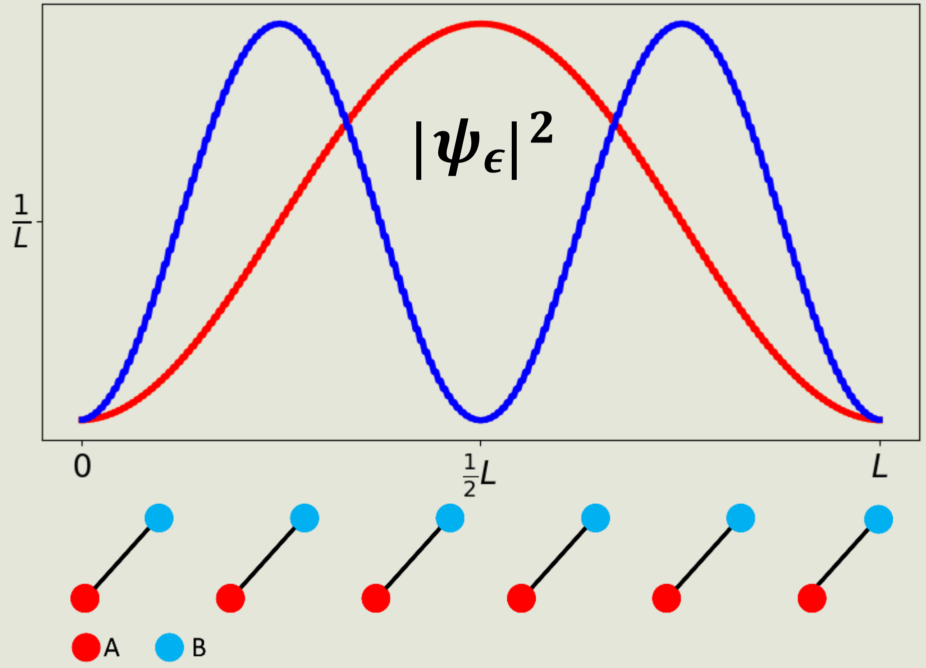

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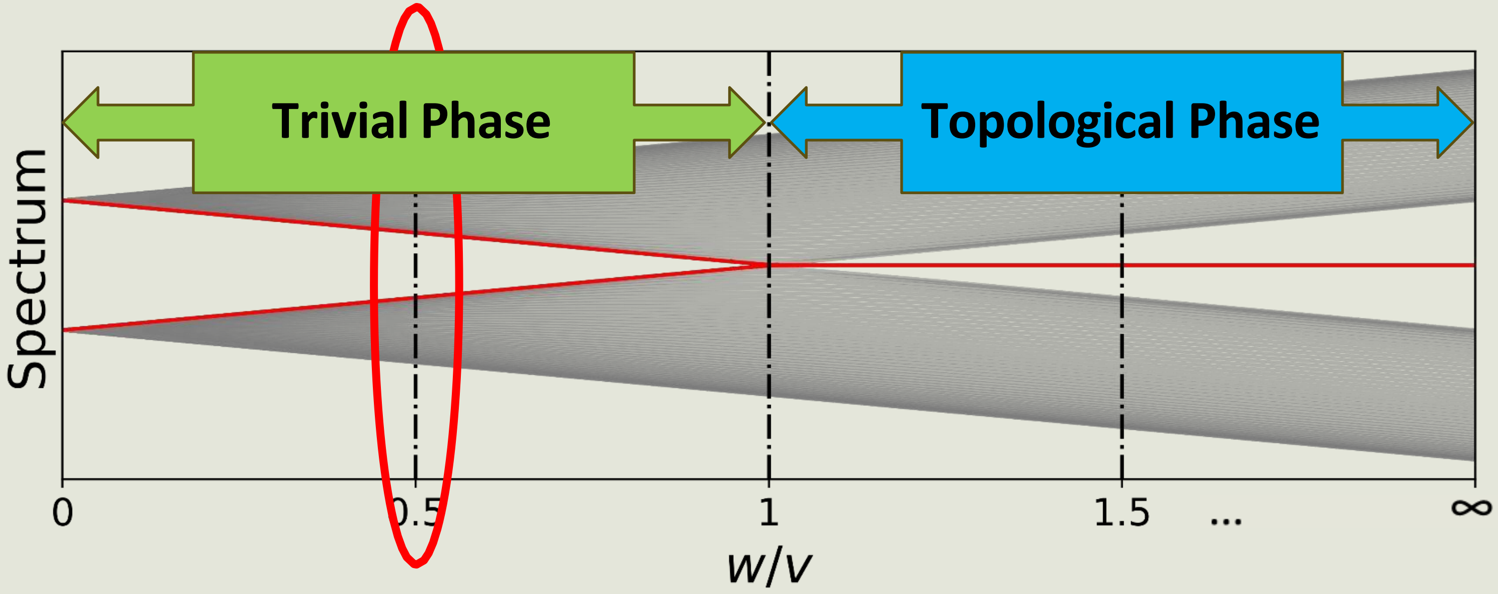

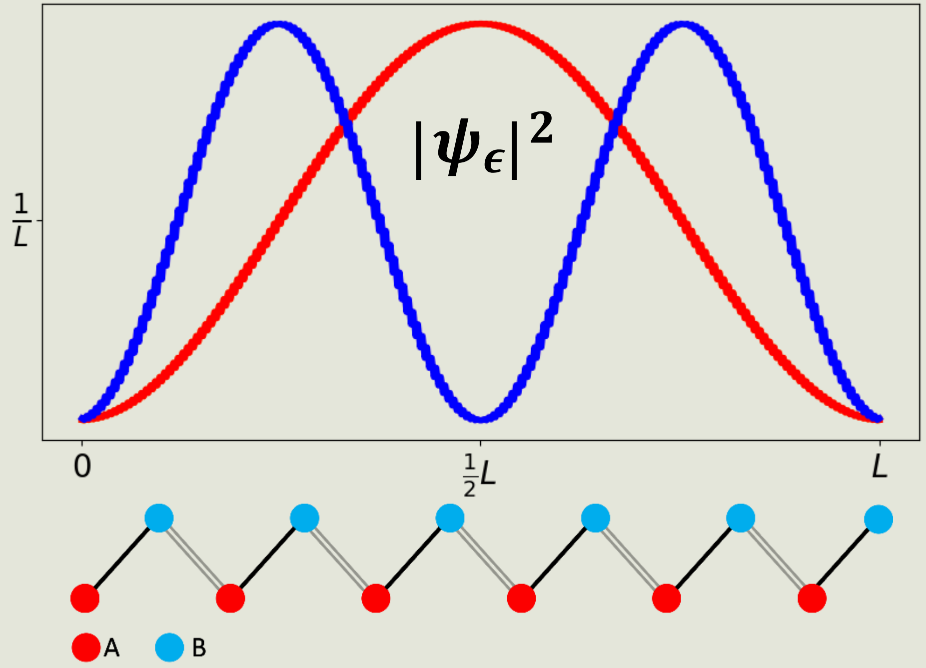

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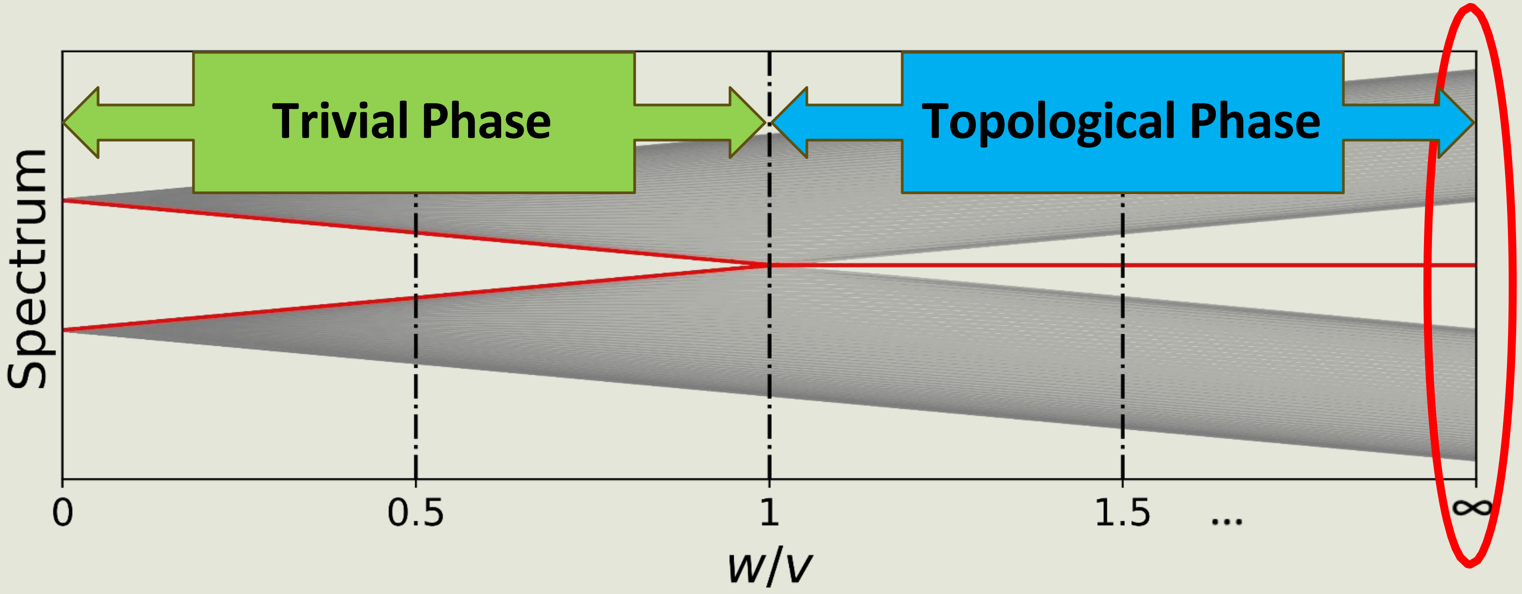

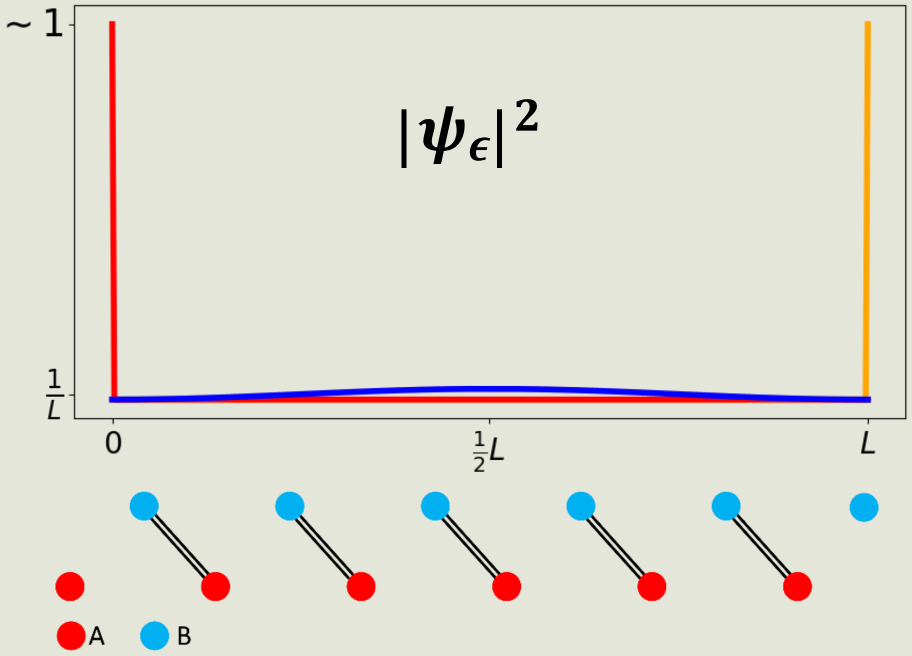

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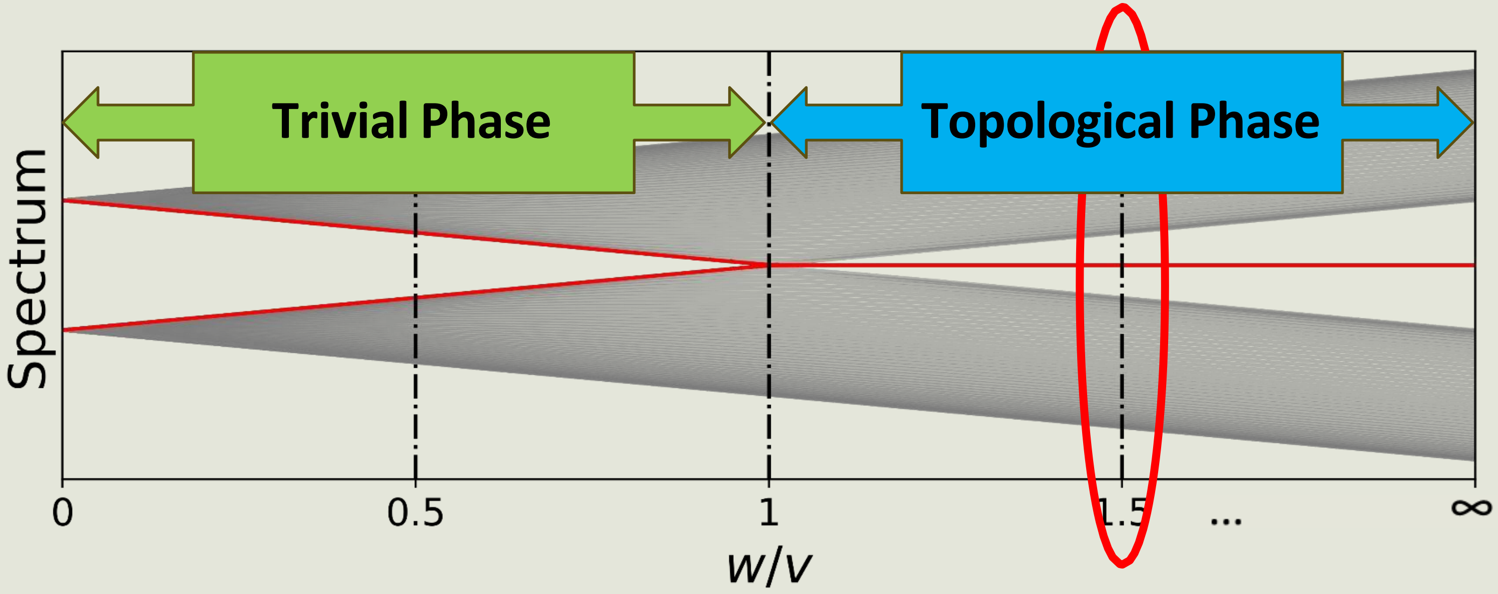
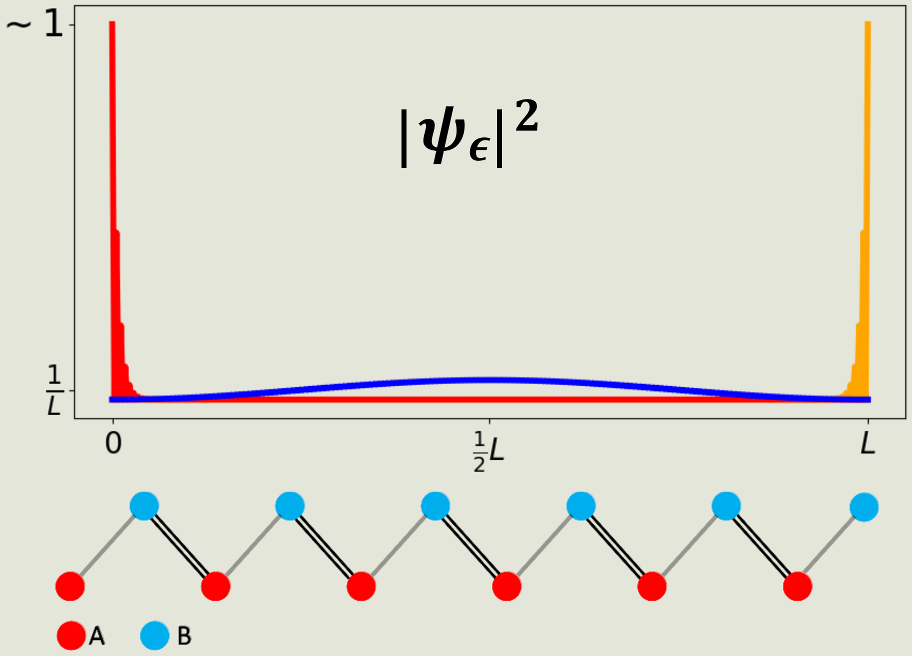




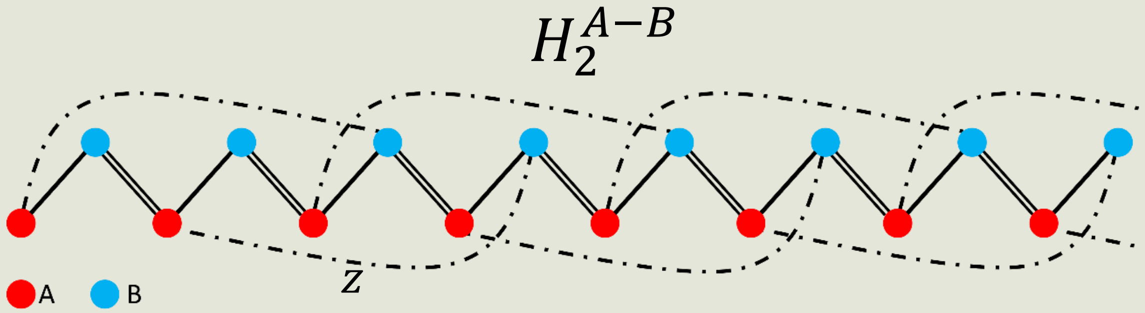
eSSH model

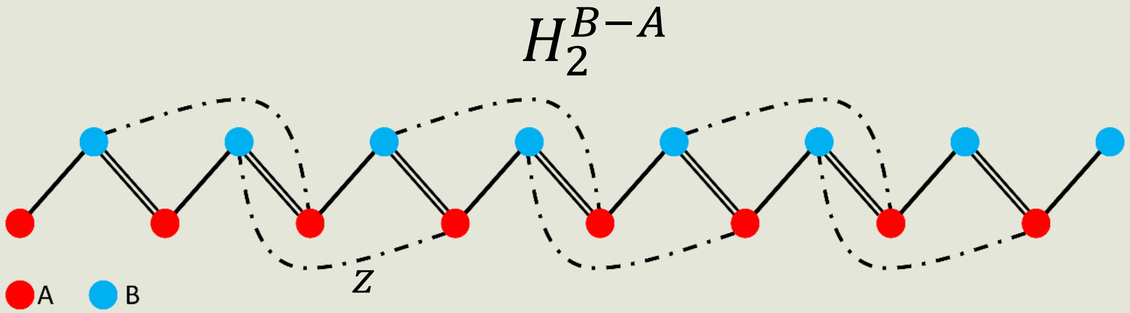
The SSH model can be extended (eSSH) by including longer range hopping at fixed dimer distance n. Accordingly, two families of eSSH can be defined:

$$H_n^{A-B} = H_{SSH} + z\sum_j c_{A,j}^{+} c_{B,j+n} + h.c.$$

$$H_n^{B-A} = H_{SSH} + z\sum_j c_{B,j}^{+} c_{A,j+n} + h.c.$$

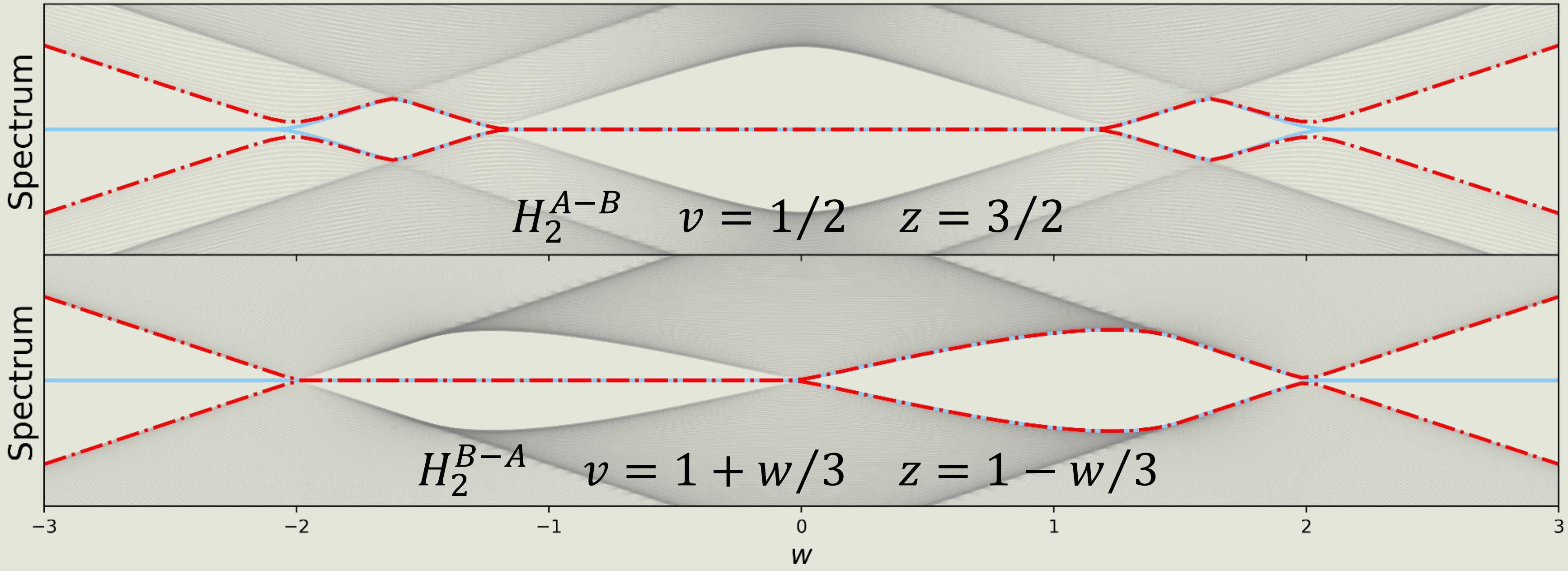
By fixing $n = 2$, the two families can be represented as:





eSSH model

As for the SSH, the eSSH spectrum is composed by two zero-symmetric bands. However, the number of topological phases is greater, as the number of edge states can range from 0 to $2n$.



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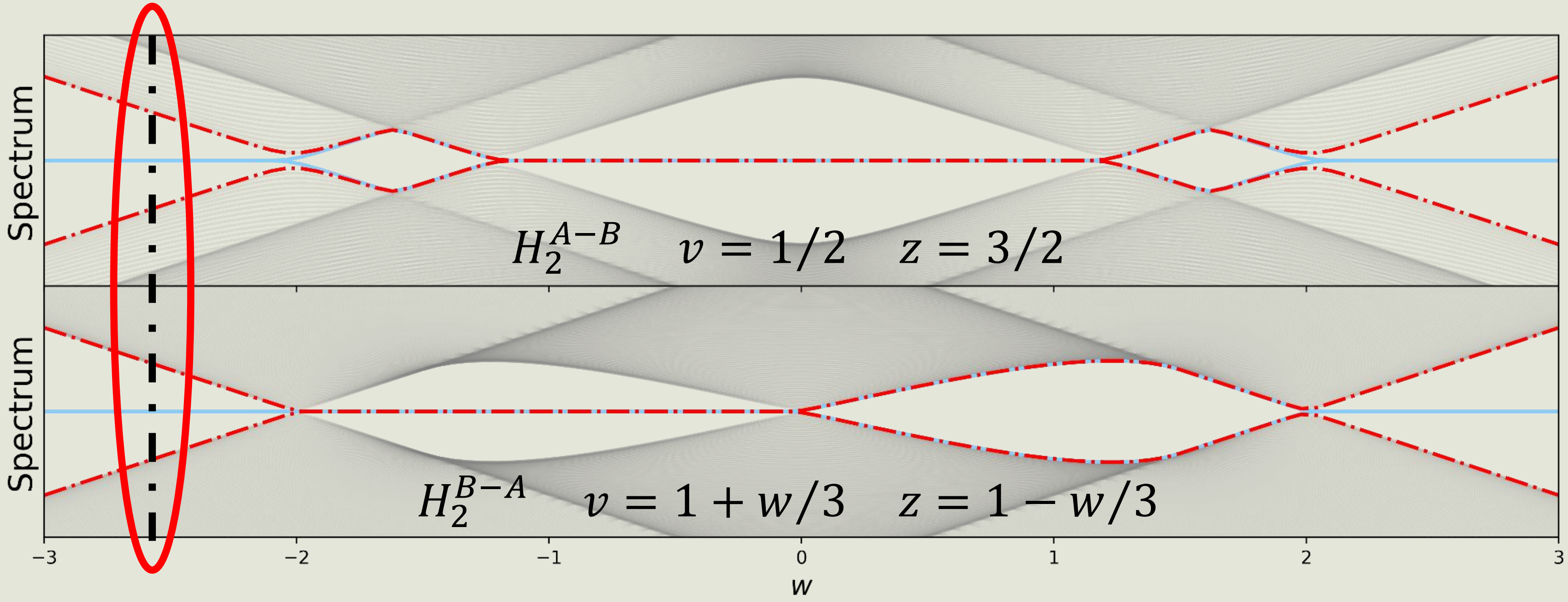

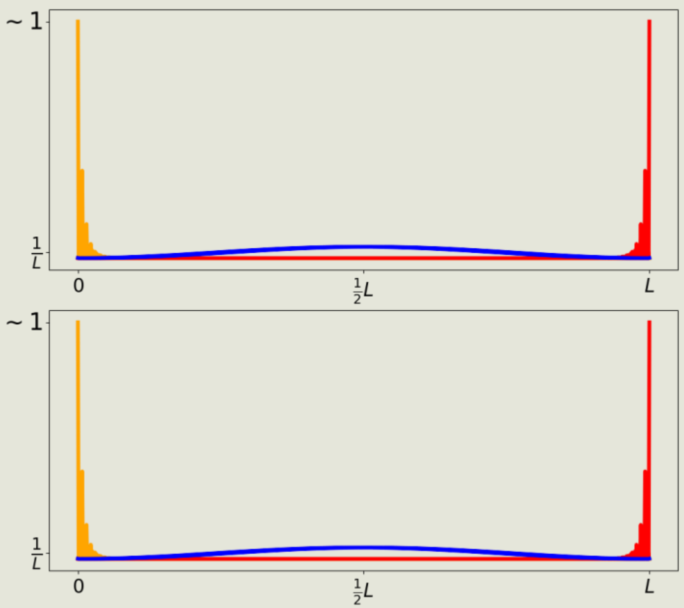

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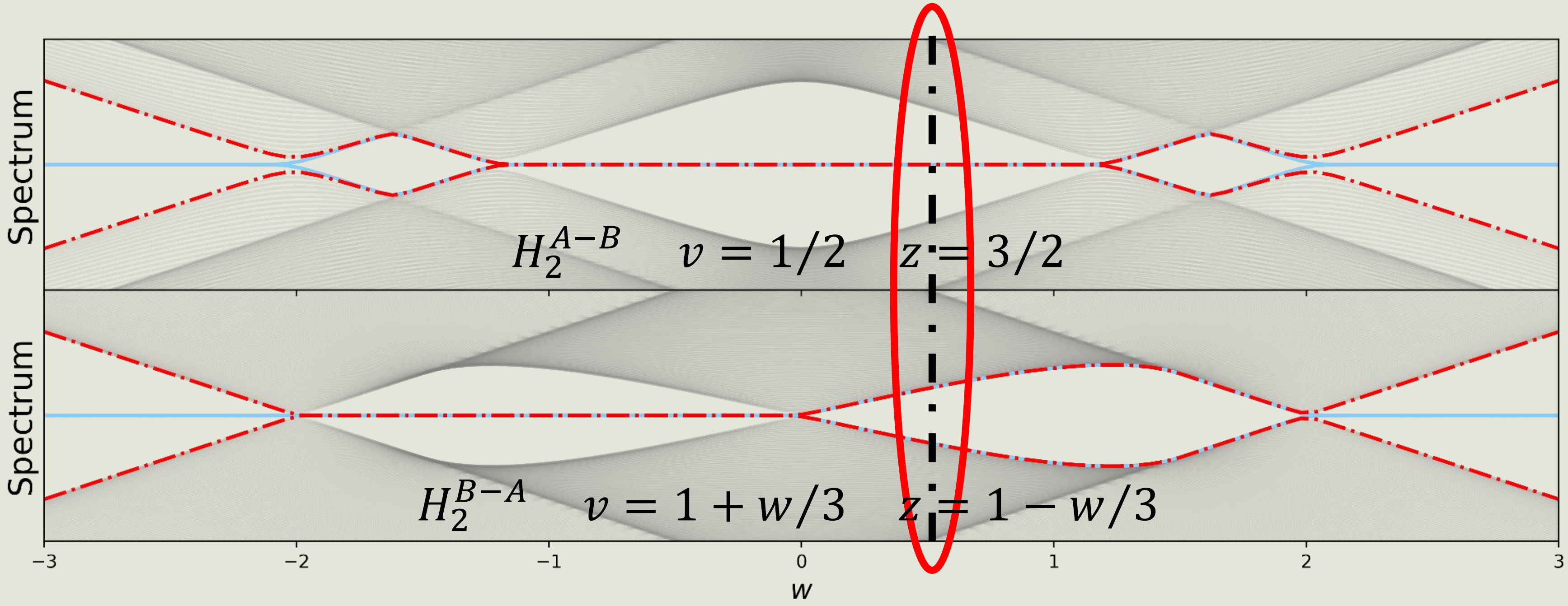
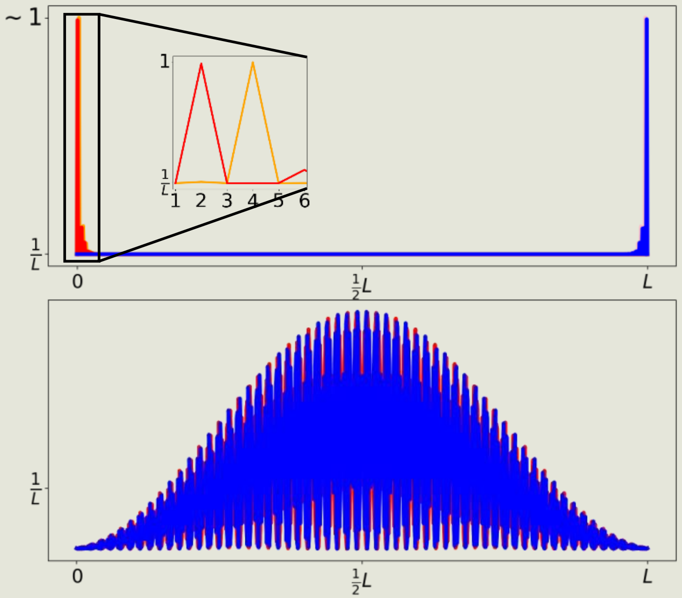




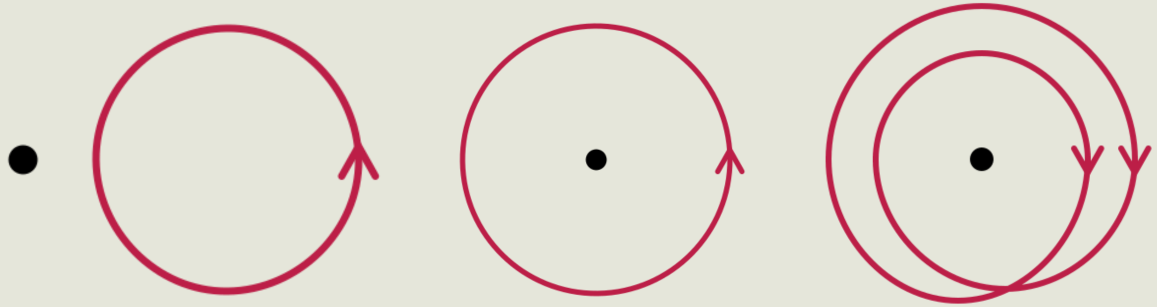
Winding Number

To count the edge states, firstly PBC are imposed and H is expressed in momentum space:

$$c_{X,k} = \frac{1}{\sqrt{L}} \sum_j e^{ikj} c_{X,j} \rightarrow H = \sum_k \left(c_{A,k}^{+}, c_{B,k}^{+}\right) h_k \begin{pmatrix} c_{A,k} \\ c_{B,k} \end{pmatrix} \qquad h_k = \vec{\gamma}_k \cdot \vec{\sigma}$$

Since eSSH possesses **chiral symmetry**, $\vec{\gamma}_k$ lies on the $x - y$ plane. Furthermore, being a closed curve, a **winding number** Ω can be defined

$$\Omega = \frac{1}{2\pi} \oint_k \frac{\gamma_x dy - \gamma_y dx}{|\gamma|^2}$$



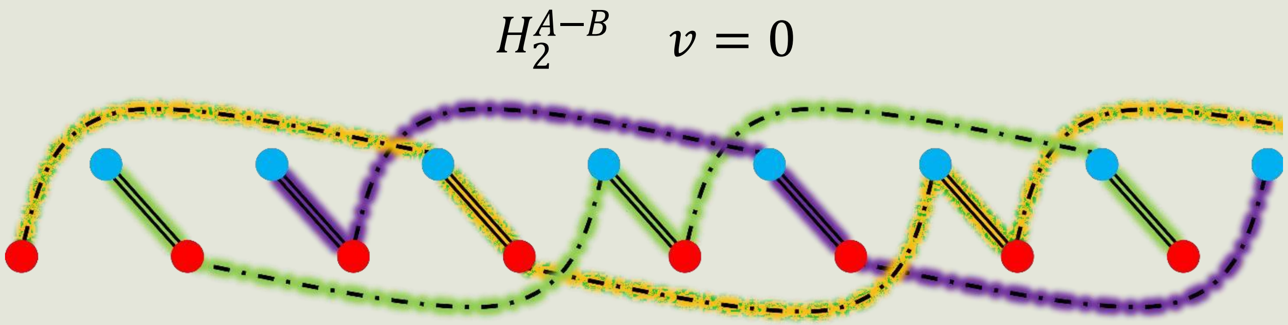
$\Omega = 0$ $\qquad$ $\Omega = 1$ $\qquad$ $\Omega = -2$

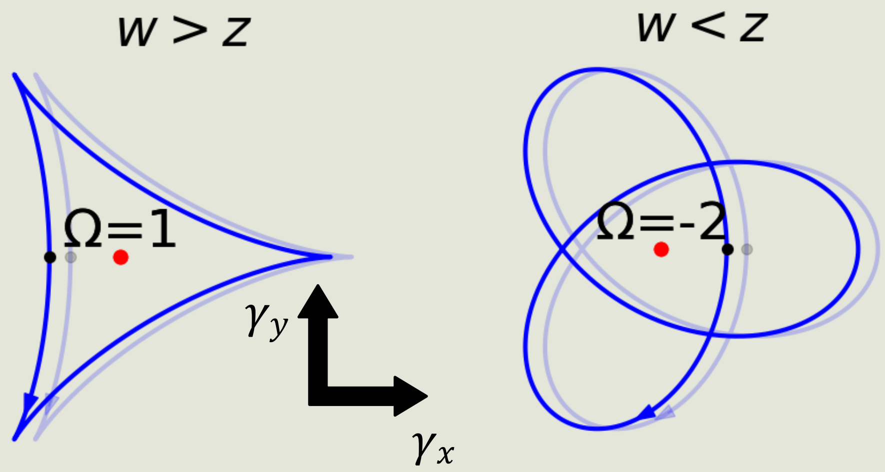
Winding Number

Then, thanks to the **bulk-boundary** correspondence, $|\Omega|$ dictates the pairs of edge states, while its sign indicates the site (A or B) of maximum localization:

- If $\Omega > 0$ then the left edge states lie on the first $|\Omega|$ sites associated to A
- If $\Omega < 0$ then the left edge states lie on the first $|\Omega|$ sites associated to B

Let's see H_2^{A-B} as an example:





Winding Number limitations

The winding number is a useful theoretical tool that allows to probe the topological phases of the system. To calculate it, the following conditions should be satisfied:

- Complete knowledge of $\vec{\gamma}_k$
- Thermodynamical limit
- H possesses chiral symmetry

In a realistic framework, these requirements may not be met.

How to bypass these limitations?

By employing the **even-odd differential occupancy** (**EOD**)!

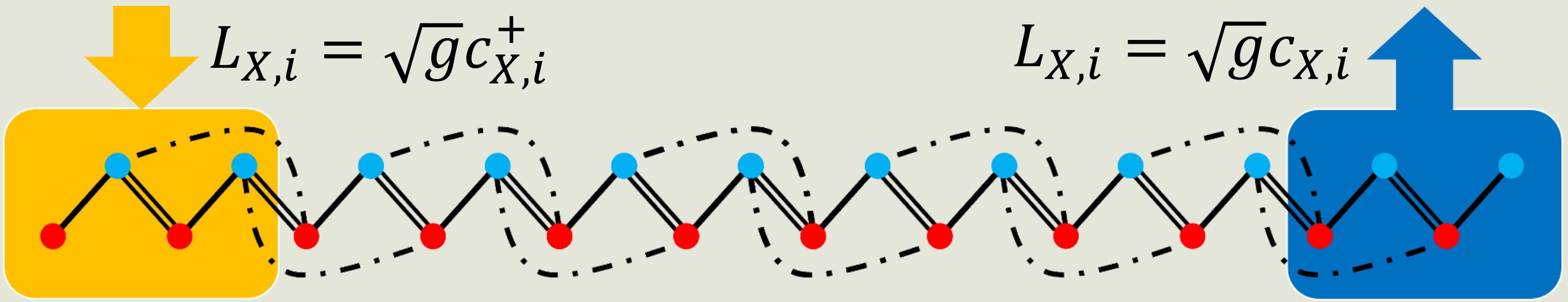
Lindblad Master Equation

If a system S can exchange particles with an external reservoir, it will reach a non-equilibrium steady state (**NESS**).

In the hypothesis of markovianity, the state ρ associated to S is governed by the Lindblad Master Equation (**LME**):

$$\dot{\rho} = -i[H,\rho] + \mathcal{L}[\rho]$$

The **NESS** is then defined as the state satisfying $\dot{\rho} = 0$. Here, we consider external reservoir injecting particles at the left and extracting them at the right side.

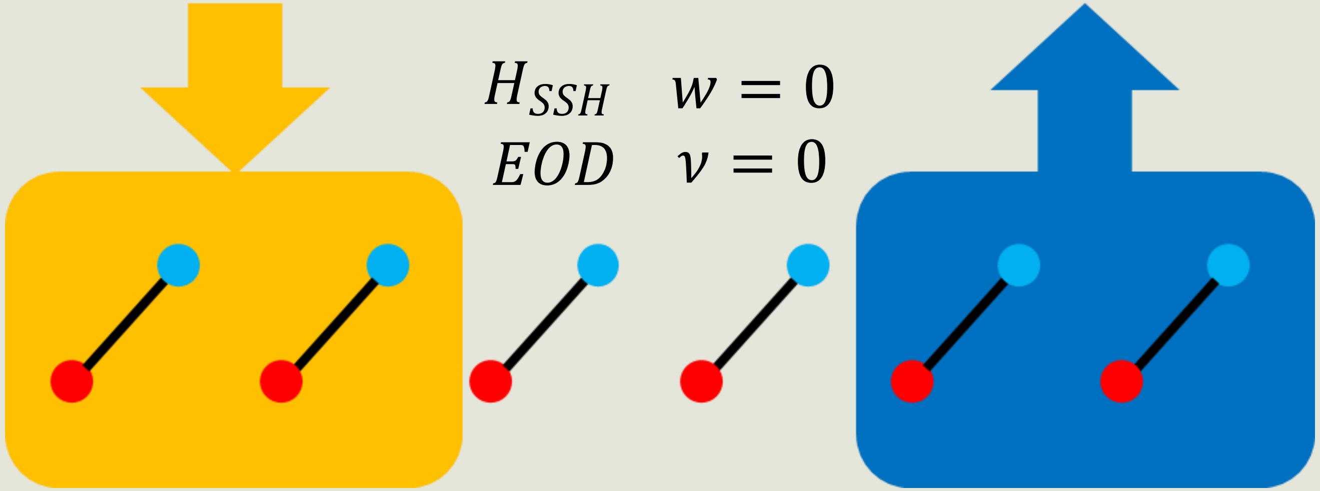


Even-odd differential occupancy

Using the state ρ, it is possible to define the EOD:

$$\nu = \sum_i (n_{A,i} - n_{B,i})$$

with $n_{X,i}$ being the average occupation. At the NESS, the EOD is a measure of imbalance of occupancy between the A and B sites, but it can also be used to probe the number of edge states.



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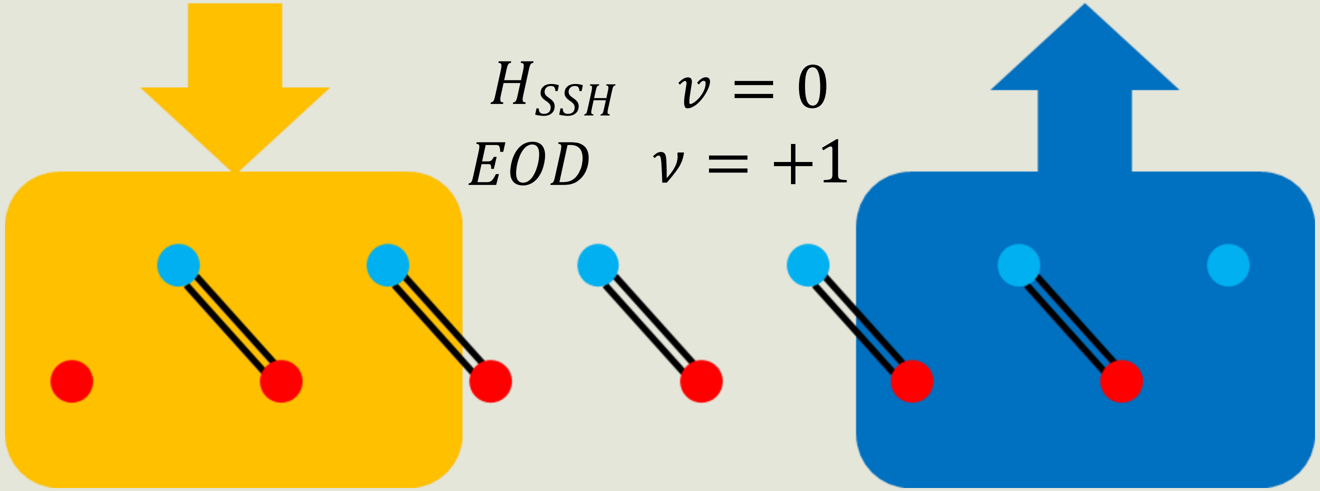
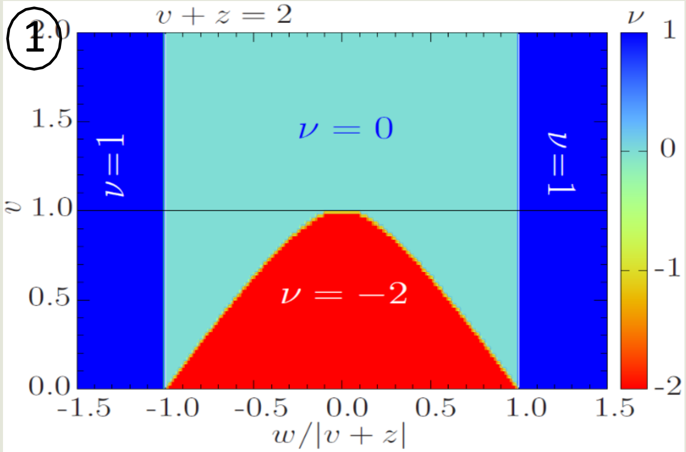
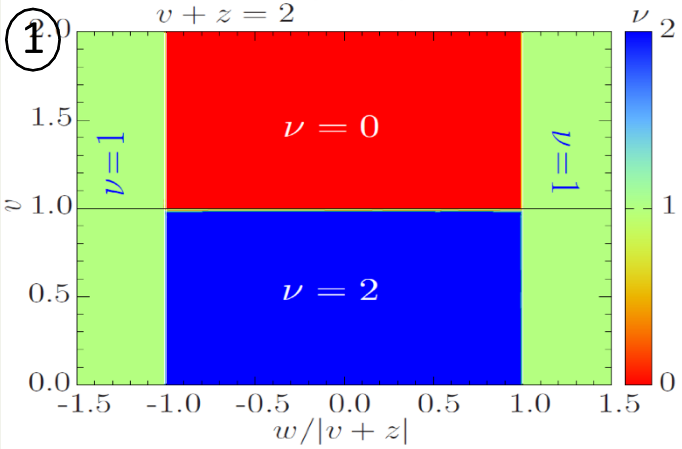


Even-odd differential occupancy

$H_2^{A-B} \Rightarrow$



$H_2^{B-A} \Rightarrow$



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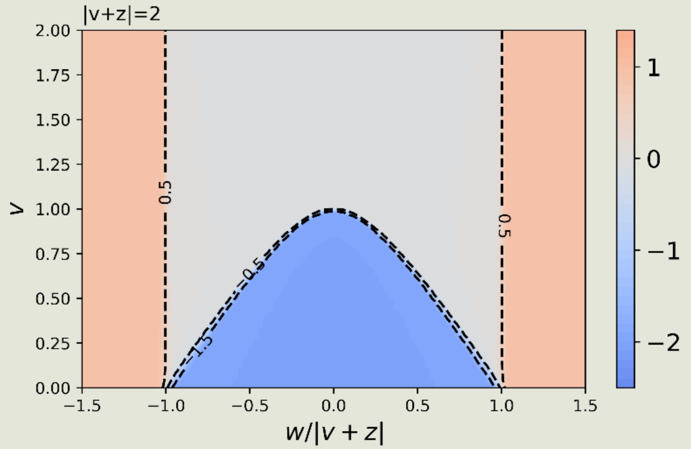
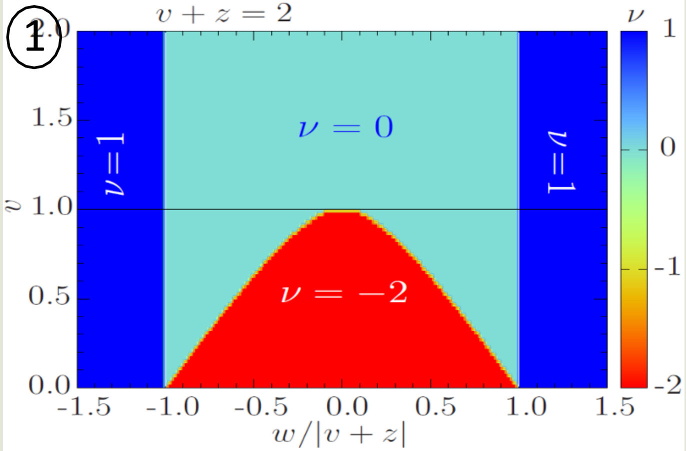
Even-odd differential occupancy

Considering the eSSH with $n = 2, \nu \sim \Omega$ with only 100 dimers!

$H_2^{A-B} \Rightarrow$





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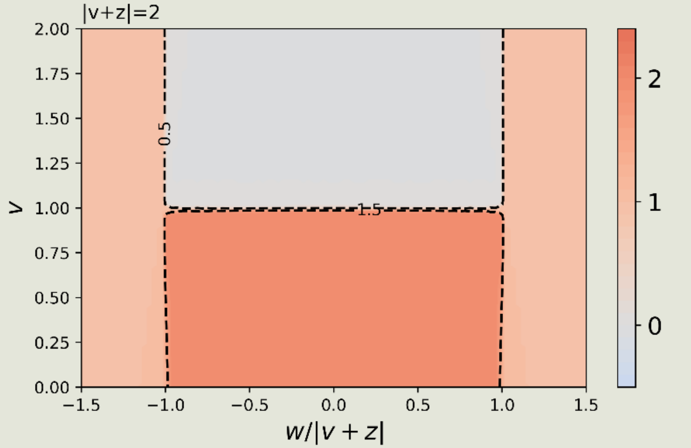

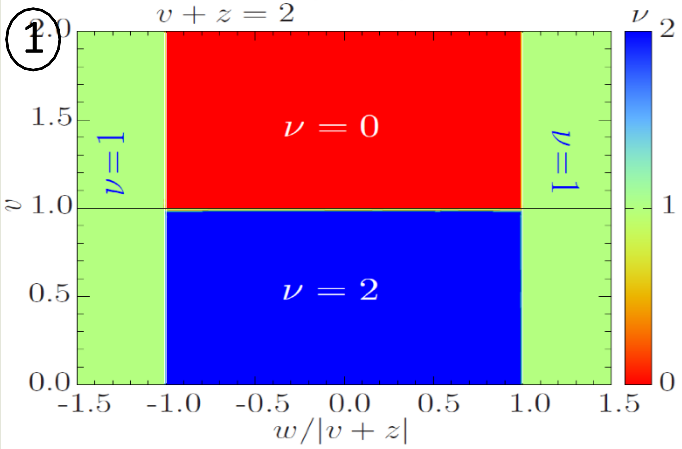

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Disordered eSSH

- Definition of disorder
- Computational steps
- Numerical Results
- Discussion of Results

Definition of disorder

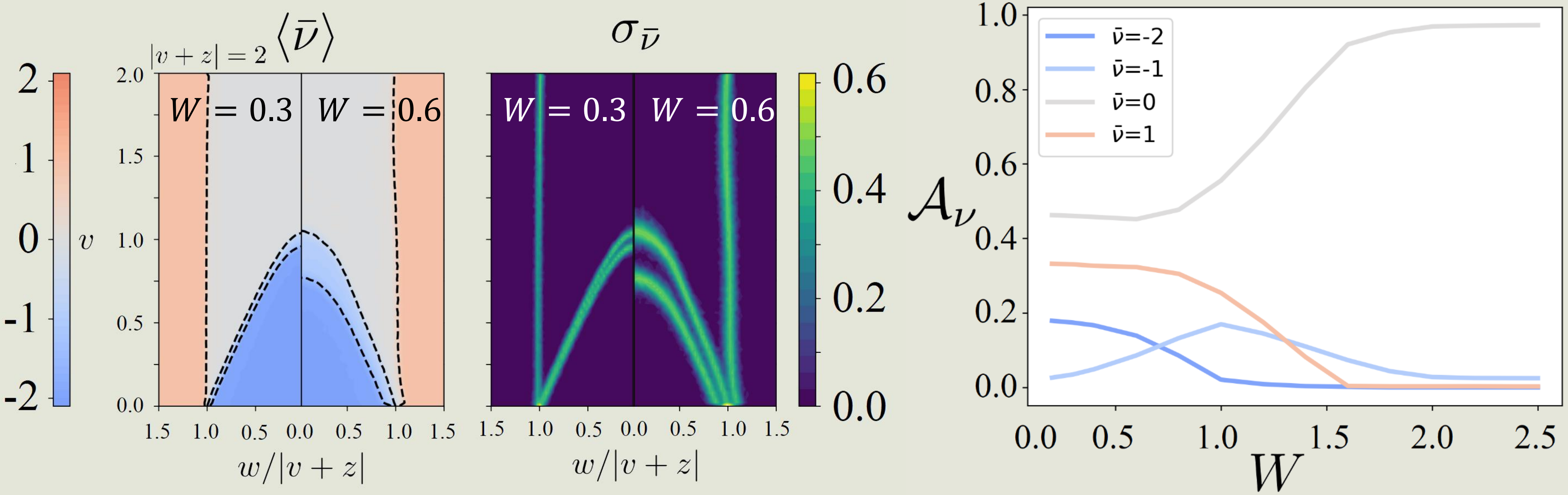
We tested the resilience of edge states against 3 types of disorder, modeled as a random perturbation:

- Type I: local **uncorrelated** perturbations to hoppings
- Type II: local **correlated** perturbations to hopping
- Type III: constant chemical potential added to a random subset of **dimers**

Type I and II disorder preserve chiral symmetry, while Type III does not.
This means that for the first two the EOD is quantized, while for the third one it doesn't.
Nonetheless, in the latter case it has been proven useful to extract information about the localization nature of the single-particle states.

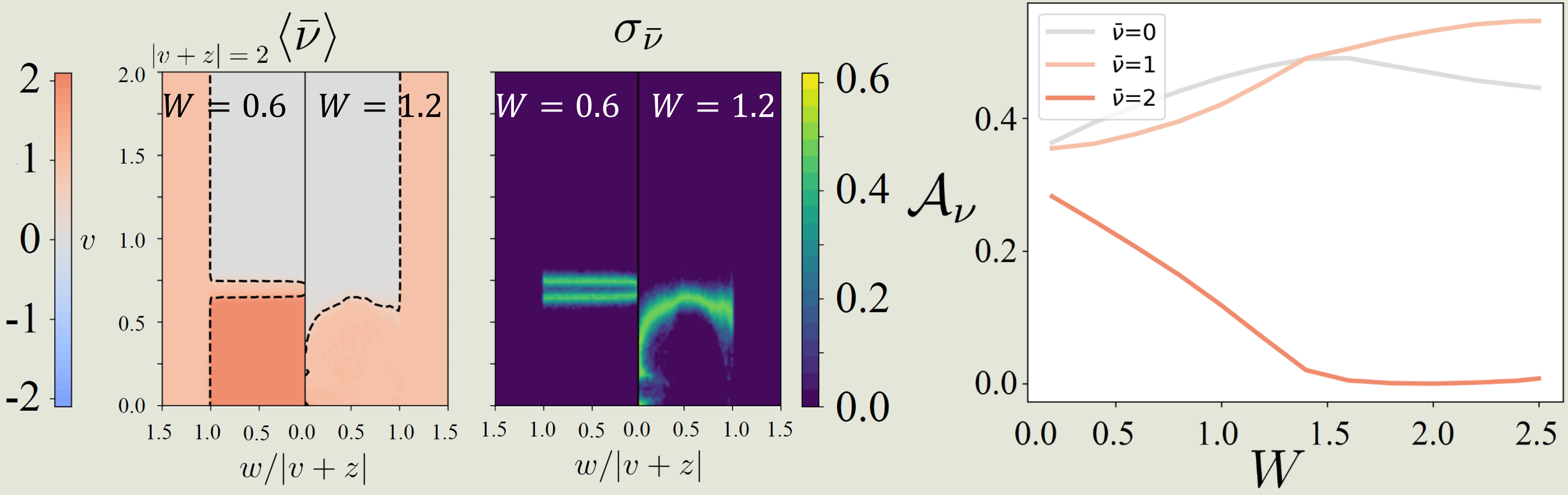
Numerical Results – Type I H_2^{A-B}

The results were produced by considering $\mathcal{N} = 400$ different disorder configurations. Calculated quantities were average $\langle \bar{\nu} \rangle$ standard deviation $\sigma_{\bar{\nu}}$ and area associated to each topological phase $\mathcal{A}_\nu$.



Numerical Results – Type II H_2^{B-A}

The results were produced by considering $\mathcal{N} = 400$ different disorder configurations. Calculated quantities were average $\langle\bar{\nu}\rangle$ standard deviation $\sigma_{\bar{\nu}}$ and area associated to each topological phase $\mathcal{A}_\nu$.



Conclusions

Conclusions

For chiral disorder (Type I & II):

1. Topological phases are resilient against disorder, and trivial regions can even become topological
2. $\sigma_\nu \sim 0$ everywhere except for phase boundaries
3. If R_1 and R_2 are two adjacent topological phases, then $\left|\langle\nu\rangle_{R_1} - \langle\nu\rangle_{R_2}\right| = 1$
4. Topological phases are destroyed in a decreasing order of $|\langle\nu\rangle|$

For non-chiral disorder:

1. For low W and far from the boundaries $\sigma_\nu \sim 0$
2. For high W EOD is no more quantized but it can be used to find whether disorder-induced localization happens preferentially on A or B sites.

Long Range Kitaev Chain

Long Range Kitaev Chain - Model

The LRK chain is a lattice model containing **next-neighbour hoppings**, **chemical potential** and **long-range pairing** terms:

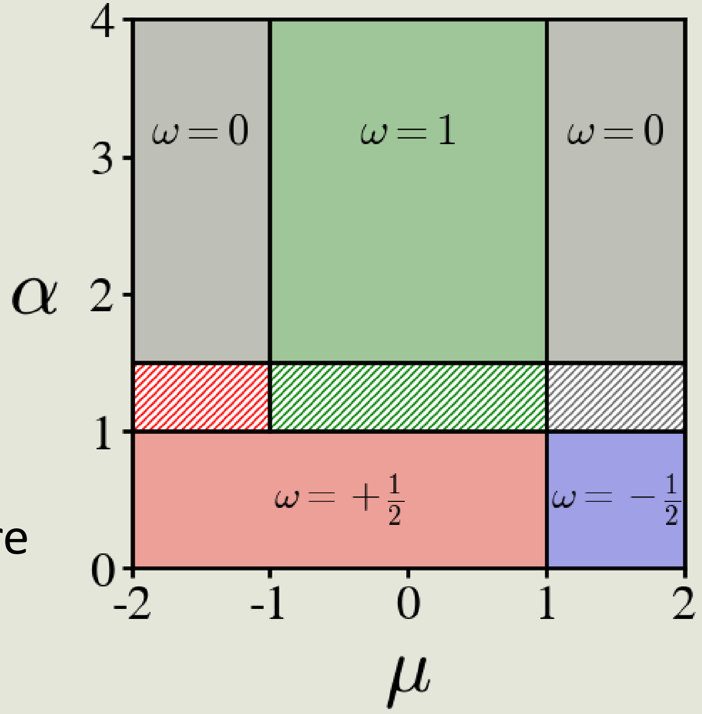
$$H_{LRK} = -w\sum_{j<L}\left(c_j^{+}c_{j+1} + c_{j+1}^{+}c_j\right) - \mu\sum_{j\le L} c_j^{+}c_j - \frac{\Delta}{2}\sum_{j<L}\sum_{r\le L-j}\frac{1}{r^{\alpha}}\left(c_j^{+}c_{j+r}^{+} + c_{j+r}c_j\right)$$

For $\alpha > \frac{3}{2}$, Ω is integer and the phase diagram is:

- $|\mu| < 1$ topological phase
- $|\mu| > 1$ trivial phase

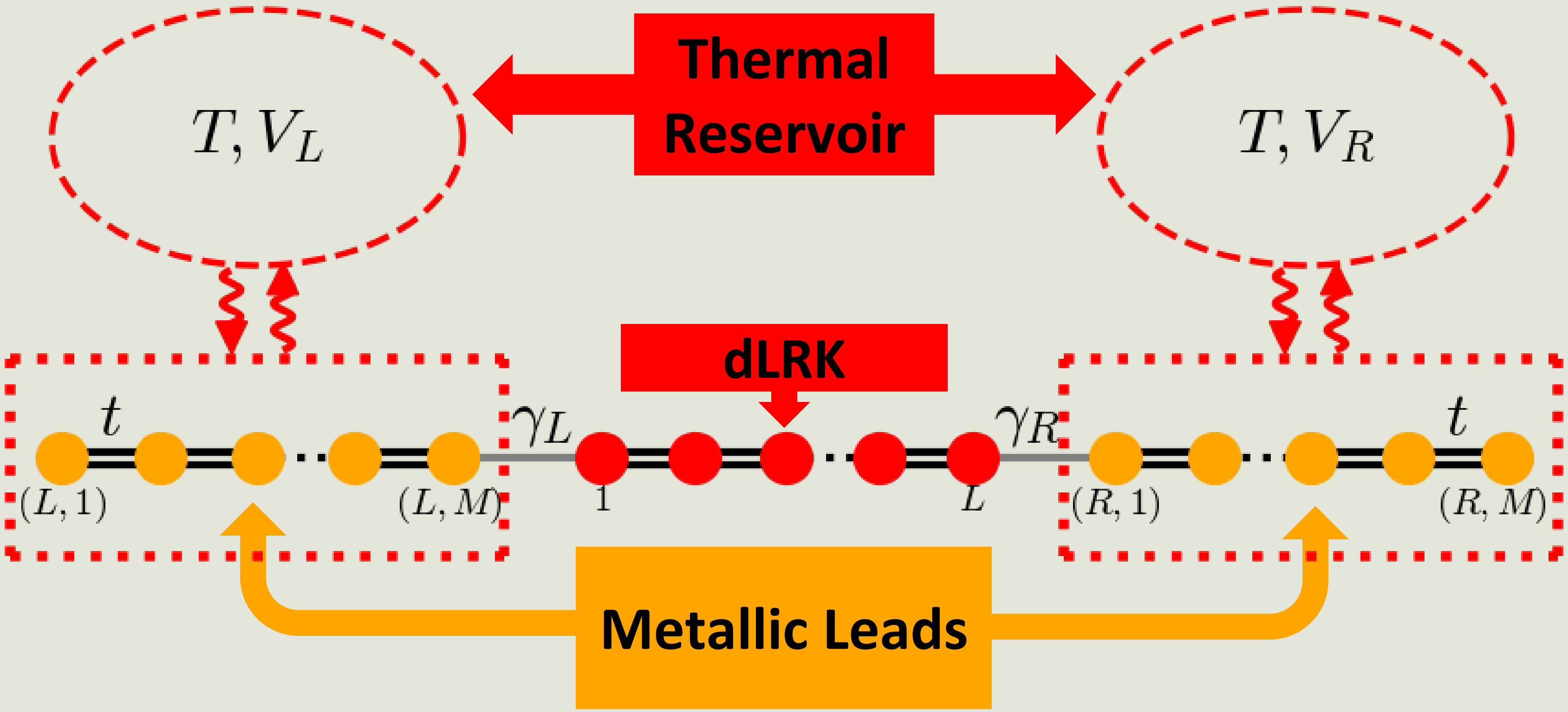
For $\alpha < \frac{3}{2}$, Ω is not quantized and for $\mu < 1$, an edge state with power-law decay and non-zero energy appears.

As for eSSH, impurity can alter LRK structure. In this work, it has been modelled as a perturbation on chemical potential (**dLRK**). The edge states are then probed through current and correlation measurements.



Long Range Kitaev Chain - Setup

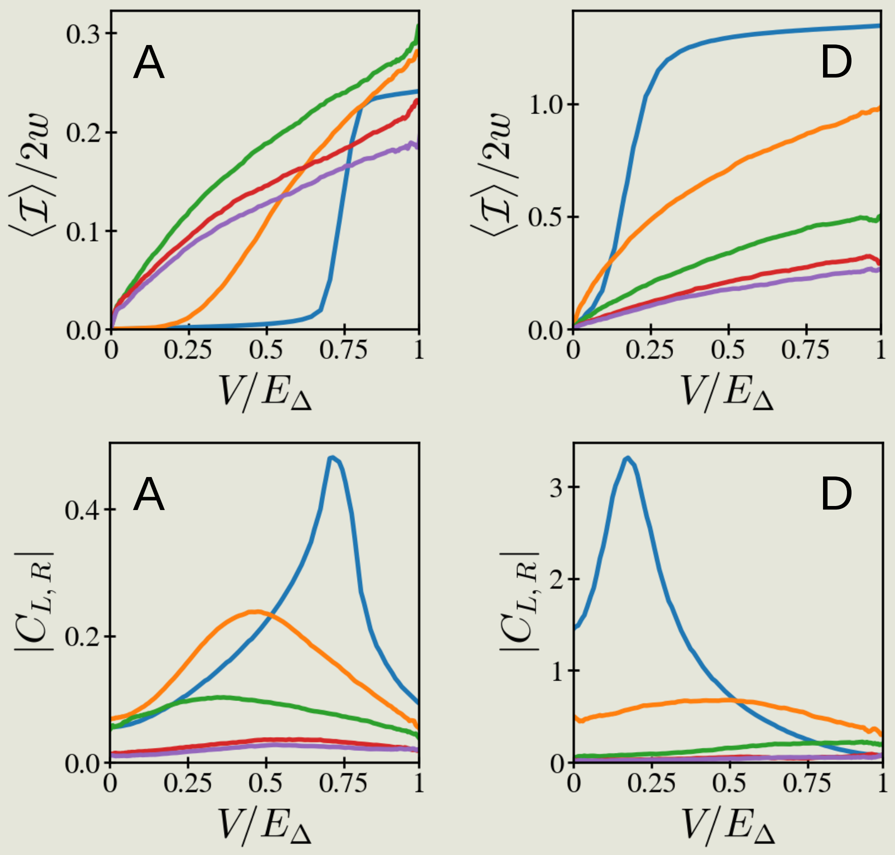
The dLRK topology was studied through the N-S-N geometry:



Long Range Kitaev Chain – Numerical Results

The numerical results show that, as for the eSSH, the topological phases are resilient to small amount of disorder. If $\langle I \rangle$ is the average lead-dLRK current and $C_{L,R}$ the lead-lead correlation:

1. $\langle I \rangle \neq 0$ and $C_{L,R}$ has a peak when $V > \epsilon_{min}$
2. In the topological phase $\langle I \rangle \gg 1$ for $V \sim \epsilon_{edge\ state}$ and:
   i. For exponentially decaying edge states $C_{L,R} \ll 1$
   ii. For power-law decaying edge states $C_{L,R}$ has a peak at $V \sim \epsilon_{edge\ state}$



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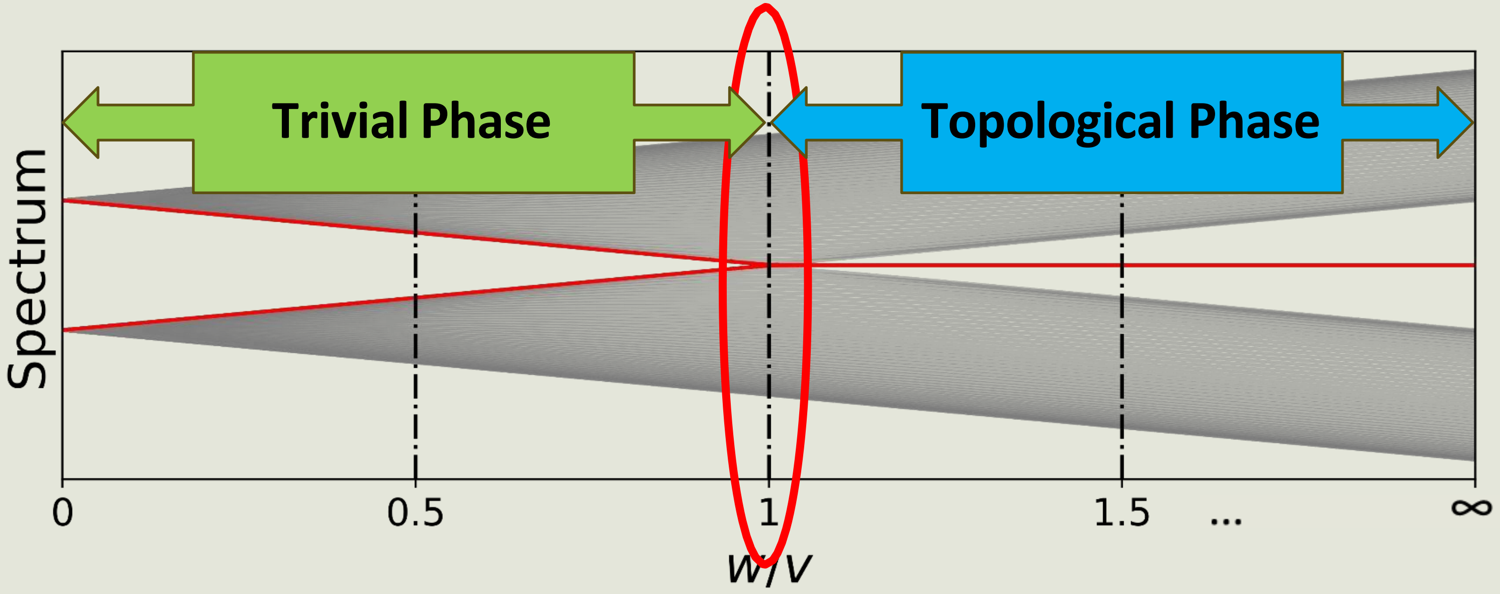
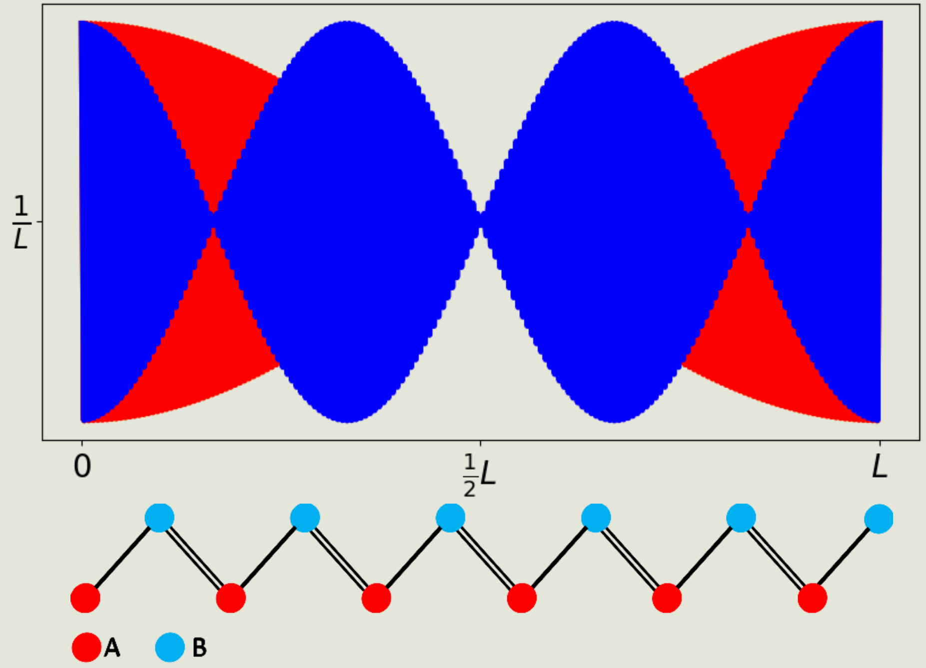
SSH model

The SSH spectrum is composed by two bands, symmetric around E=0. For w>v the spectrum contains two zero-energy modes, exponentially localized at the two edges.





Definition of disorder

To test the resilience of edge states against disorder, it was modelled as random local perturbations on H. Both chiral and non-chiral disorder were considered; the first one is taken as a local offset on v, w and z:

$$v_i = v + \epsilon_{1i} \qquad w_i = w + \epsilon_{2i} \qquad z_i = z + \epsilon_{3i}$$

Two types of chiral disorder were chosen:

- Type I (uncorrelated): $\{\epsilon_{ji}\}$ comes from a uniform distribution such that $\langle\epsilon\rangle = 0$ and $\sigma_\epsilon = W$
- Type II (correlated): $\{\epsilon_{ji}\}$ is chosen in two different ways:
  - $n = 2 \Rightarrow \epsilon_{2i} = 0$, while $\epsilon_{3i} = -\epsilon_{1i}$ are drawn from a binomial distribution having $\epsilon \in \{0, W\}$ with 50% chance
  - $n = 3 \Rightarrow \epsilon_{1i} = 0$, while $\epsilon_{3i} = -\epsilon_{2i}$ are drawn from a binomial distribution having $\epsilon \in \{0, W\}$ with 50% chance

Conversely, the non-chiral disorder is chosen as chemical potential term added to a random subset of dimers with 50% chance (Type III):

$$H_W = W \sum_{i \in R} \left(c_{A,i}^{+} c_{A,i} + c_{B,i}^{+} c_{B,i} \right)$$

Even-odd differential occupancy

Starting from ρ, it is possible to define the EOD:

$$\nu = Tr(\Gamma\rho) = \sum_i \left(C_{A,i;A,i} - C_{B,i;B,i}\right)$$

with $C_{X,i;Y,j} = Tr(c^{+}_{X,i} c_{Y,j} \rho)$ being the correlation matrix. Since H is quadratic and $\{L_{X,i}\}$ are linear, C admits a closed set of differential equations, defined starting from LME:

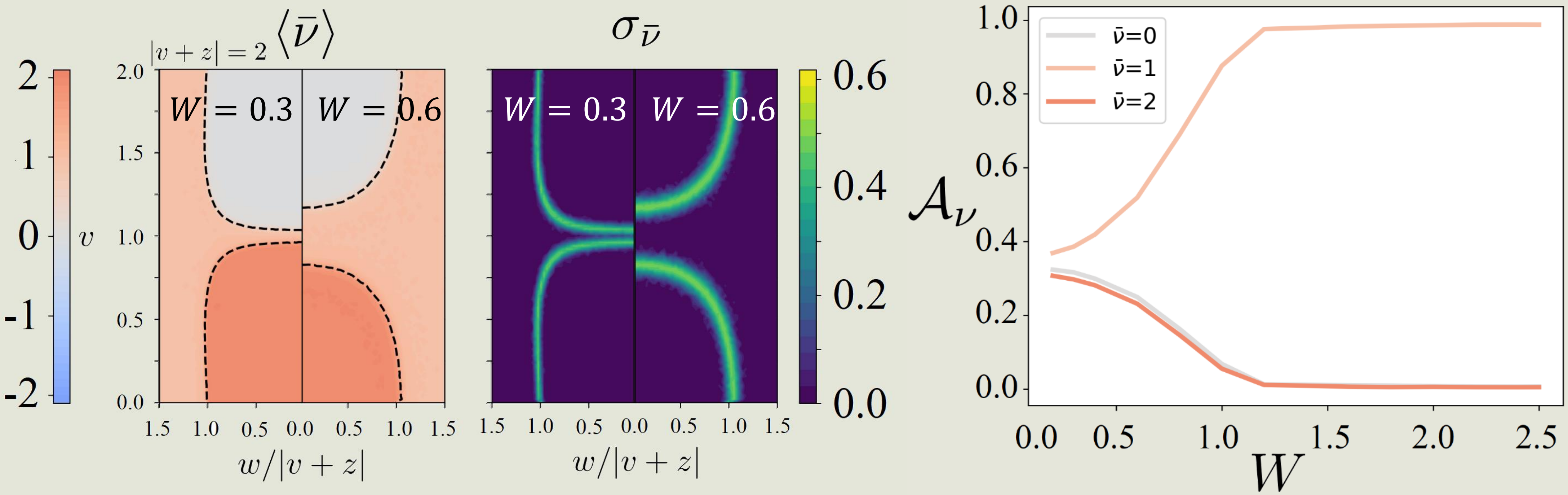
$$\dot{C} = i[\mathcal{H}^T, C] - \frac{1}{2}\{\mathcal{G} + \mathcal{R}, C\} + \mathcal{G}$$

where $\mathcal{H}$ is the matrix of coefficients of H, while $\mathcal{G}$ and $\mathcal{R}$ are diagonal matrices encoding the interactions with the jump operators.

Since at the NESS $\dot{\rho} = 0$, then $\dot{C} = 0$; consequently, the differential equation reduces to a system of linear equations.

Numerical Results – Type I H_2^{B-A}

The results were produced by considering $\mathcal{N} = 400$ different disorder configurations. Calculated quantities were average $\langle \bar{\nu} \rangle$ standard deviation $\sigma_{\bar{\nu}}$ and area associated to each topological phase $\mathcal{A}_\nu$.



Numerical Results – Type III H_2^{B-A}

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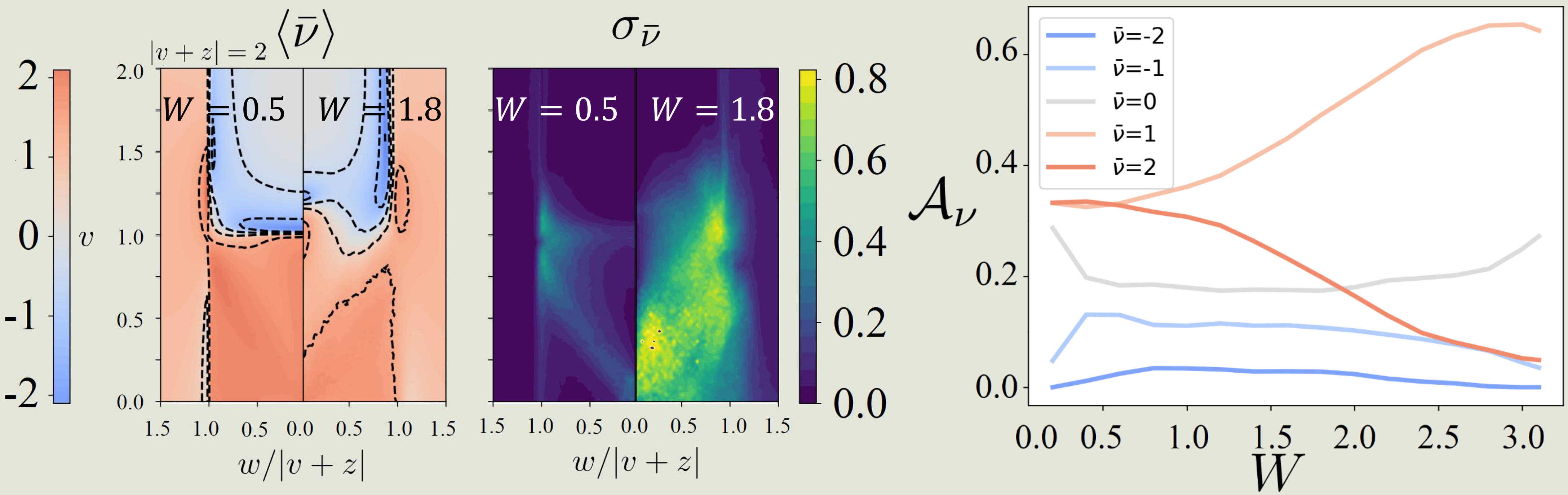
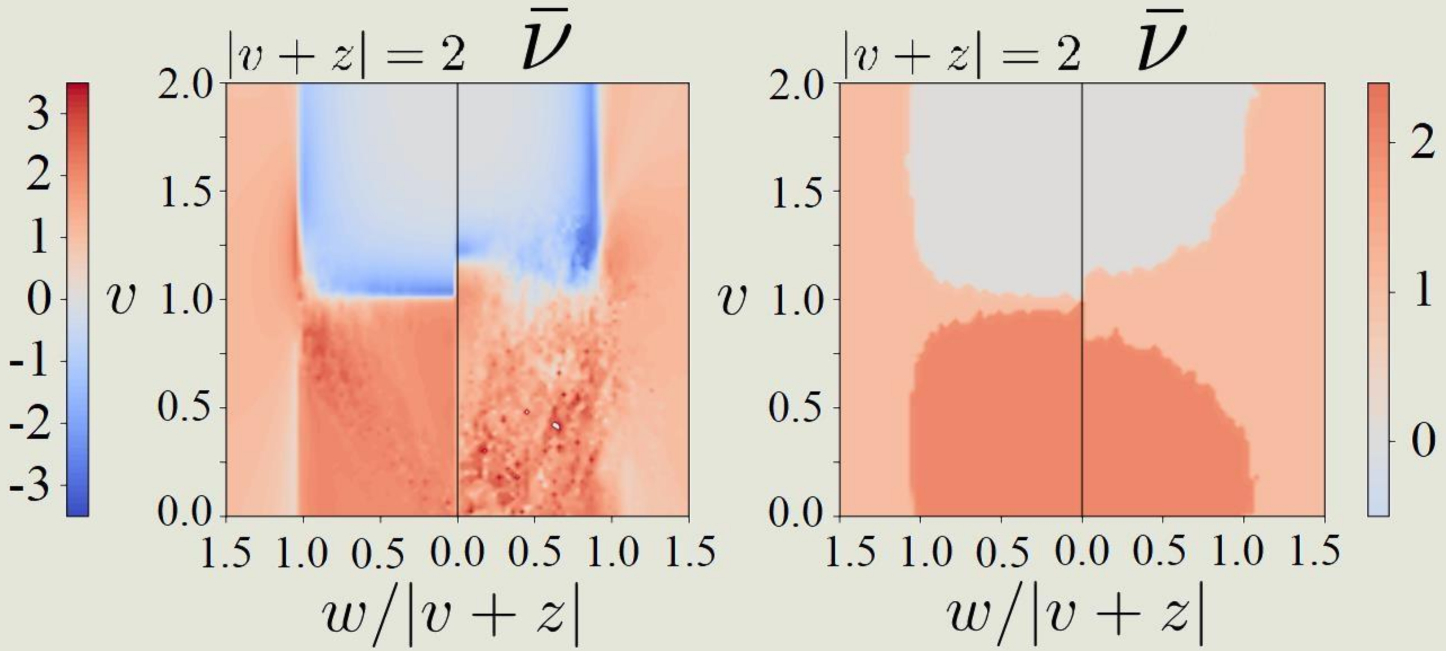


Discussion of results – Type III H_2^{B-A} EOD quantization

Since Type III disorder violates chiral symmetry, EOD is no more quantized. This can be noticed by comparing Type I and Type III EOD for a single disorder configuration:



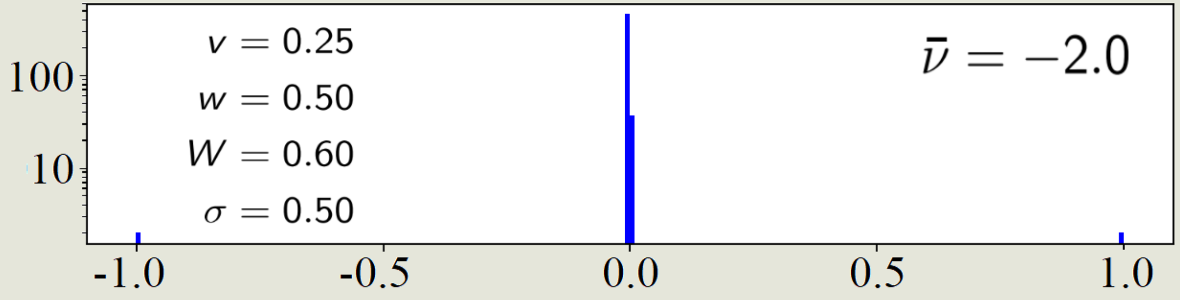
Nonetheless, for low W topological regions survives and in fact EOD is approximately quantized. Conversely, for high W disorder washes out every phase.

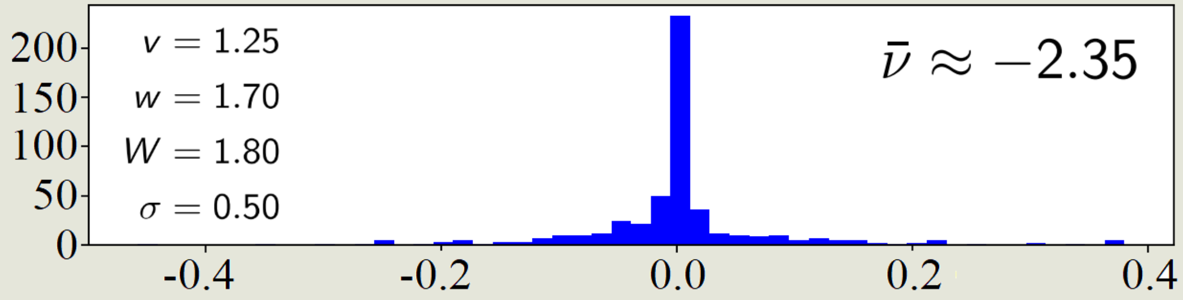
Discussion of results – Physical meaning of EOD

How do EOD values arise? To answer this question, let's define the single particle EOD associated to the Hamiltonian eigenmodes, i.e. $\gamma_\epsilon = \sum \psi_{\epsilon,j} c_j$ such that $H = \sum_\epsilon \epsilon \gamma_\epsilon^+ \gamma_\epsilon$:

$$\nu_\epsilon = \sum_j (-1)^{j+1} \left|\psi_{\epsilon,j}\right|^2$$

Then, the distribution of EOD values for chiral and non-chiral disorder looks quite different:



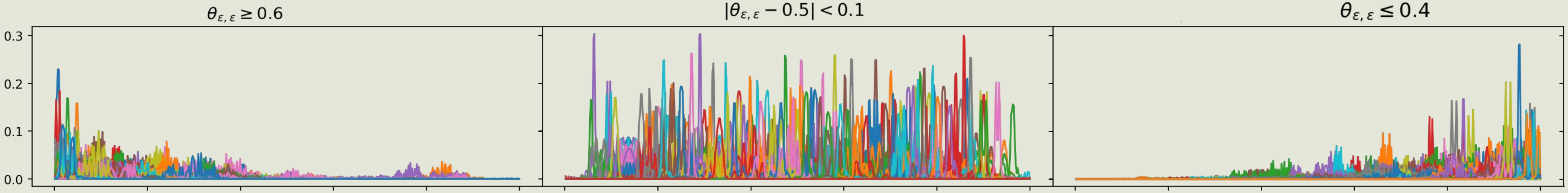


Finally, we numerically checked that the total EOD is well approximated by the weighted sum of ν_ϵ:

$$\nu \approx \sum_\epsilon \theta_\epsilon \nu_\epsilon \qquad \text{with} \qquad \theta_\epsilon = \sum_{m,n} \psi_{\epsilon,m} C_{mn} \psi^*_{\epsilon,n}$$

Discussion of results – Physical meaning of EOD

Lastly, to investigate the connection between EOD and $\psi_{\epsilon,j}$ localization for Type III disorder, we first group them by θ_ϵ:



- Higher θ_ϵ ⇔ Lower right tail
- Lower right tail ⇔ negative (but not quantized) EOD

Finally, no edge state has been found, but states having different tails and peaks position in the chain.

From all these information, the conclusion is that EOD measures the eigenstate tendency to lie on the A or on the B site of the chain, rather than counting the number of edge states.

Winding Number

Specifically, $|\Omega|$ dictates the number of edge states pairs, while its sign indicates the site of maximum localization:

- If $\Omega > 0$ then the left (right) edge states lie on the first (last) Ω sites associated to A (B)
- If $\Omega < 0$ then the left (right) edge states lie on the first (last) Ω sites associated to B (A)

Let's see H_2^{A-B} as an example:

H_2^{A-B} $\quad w = 0$